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Pivotal methods in the propagation of distributions

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Abstract
We propose a method for assigning a probability distribution to an input quantity. The distribution is used in a Monte Carlo method described in Supplement 1 to the *Guide to the Expression of Uncertainty in Measurement* for uncertainty evaluation. The proposed method provides an alternative to other methods, such as the principle of maximum entropy and Bayesian procedure that were used in Supplement 1 for the same purpose. The method is based on an exact or approximate pivotal quantity and is easily applied. We use several examples from commonly known models to illustrate the implementation of the proposed approach.

1. Introduction

Supplement 1 to the *Guide to the Expression of Uncertainty in Measurement* [1] advocates the use of the propagation of probability distributions as a basis for the evaluation of uncertainty of measurement. The probability distribution for a measurand θ is obtained by propagating the probability distributions of input quantities $\mu_1, \mu_2, \ldots, \mu_k$ that relate to θ through a known functional relationship of the form

$$\theta = f(\mu_1, \mu_2, \ldots, \mu_k).$$

An important step in the Supplement 1 approach is the *assignment* of the joint probability distribution of μ_i to be used in the propagation. The assignment is based on the available information on μ_i. Supplement 1 provides a list of probability distributions assigned to input quantities on the basis of the available information [1, section 6.4]. For example, if the only available information regarding an input quantity μ is a lower limit a and an upper limit b (both a and b are known), then, by use of the principle of maximum entropy, Supplement 1 would assign a uniform distribution, uniform(a, b), to μ. If the information about μ consists of repeated measurements from a known distribution with unknown parameters, Supplement 1 uses Bayes' rule to calculate a probability distribution for μ. For example, if $X_1, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$, by selecting a non-informative joint prior distribution for μ and σ, and using Bayes' theorem, Supplement 1 assigns a scaled and shifted Student's t distribution to μ.

In many applications, the available information on an input quantity for probability distribution assignment is different from the information used in Supplement 1. Suppose both a and b in the above example are unknown, but repeated measurements from uniform (a, b) are available. If the desired input quantity μ_0 is taken to be equal to $(a + b)/2$, the mean of the uniform distribution, then what is the probability distribution that should be assigned to μ_0? Both Bayesian and fiducial methods [2] may be used to answer this question. In this paper we propose an alternative method, called *pivotal* method, that can also be used to assign a probability distribution for μ_0. In general, the paper is concerned with assigning a probability distribution to an input quantity μ, where μ is a function of unknown parameters (e.g. $(a+b)/2$) of a known model (e.g. the uniform distribution), and where repeated measurements are available for estimating the model parameters.

The proposed method uses a pivotal quantity of μ to derive the probability distribution. When an 'exact' pivotal quantity of μ does not exist, approximate or asymptotic pivotal quantities can be used. The pivotal method is not new and has been used to solve other inference problems [3]. The method is closely related to the concept of confidence distribution (CD). The CD has a long history and there is a renewed interest in

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this subject. For the history and recent developments on CD, see, for example, [4, 5] and references listed therein.

The rest of the paper is organized as follows. Section 2 discusses the pivotal method in detail. In section 3 we work out several commonly used probability distributions, such as normal, exponential, uniform and triangular to illustrate the method. The fiducial method, which also associates a probability distribution with μ, may be viewed as a generalization of the pivotal method. We compare these two procedures and conclude with some summary remarks in section 4.

2. Pivotal method

We briefly review the concept of a pivotal quantity for a parameter. Let $\boldsymbol{X}$ denote an observable random vector whose distribution is indexed by a (possibly vector) parameter $\boldsymbol{\xi}$. A pivotal quantity, say, $Q(\boldsymbol{X}, \boldsymbol{\xi})$ is a function of $\boldsymbol{X}$ and $\boldsymbol{\xi}$ such that the distribution of Q is free of model parameter $\boldsymbol{\xi}$. Suppose θ is a function of $\boldsymbol{\xi}$. We say that Q is a pivotal quantity for θ if Q is a pivotal quantity with the additional property that when the observed (realized) value $\boldsymbol{x}$ is substituted for $\boldsymbol{X}$, the observed value q of Q depends only on θ. For example, consider a random sample $X_1, \ldots, X_n$ from a $N(\mu, \sigma^2)$ distribution. Here, $\boldsymbol{\xi} = (\mu, \sigma)$. It is known that

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \tag{1}$$

is distributed as $N(0, 1)$, where $\bar{X}$ is the sample mean. The distribution of Z is free of unknown parameters μ and σ. That is, Z is a pivotal quantity for $\boldsymbol{\xi}$. Since the observed value of Z depends on σ, Z is a pivotal quantity for μ only if σ is known. For the case of unknown σ, we consider the statistic

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}, \tag{2}$$

where S is the sample standard deviation. It is known that T is distributed as a Student's t distribution with $n - 1$ degrees of freedom. Also, the observed value of T, $t = \sqrt{n}(\bar{x} - \mu)/s$, depends only on μ. Thus, T is a pivotal quantity for μ.

The pivotal method assigns a probability distribution for μ based on the pivotal quantity for μ. Let $Q(\boldsymbol{X}, \mu)$ be a pivotal quantity for μ. Write

$$Q(\boldsymbol{X}, \mu) = D. \tag{3}$$

Then the distribution of D is free of any unknown parameters. If Q is invertible as a function of μ for each fixed $\boldsymbol{X}$, then the pivotal method assigns the probability distribution for μ as the probability distribution of the random variable

$$Q^{-1}(\boldsymbol{x}, D), \tag{4}$$

where Q^{-1} is an inverse image of the function Q. That is, we solve for μ in (3) and substitute $\boldsymbol{X}$ with its realized value $\boldsymbol{x}$ in the solution. The assigned probability distribution for μ is the probability distribution of the solution of μ.

For the example of $N(\mu, \sigma^2)$ with known σ, if we solve for μ in (1) and replace $\bar{X}$ with $\bar{x}$ in the solution, then we obtain

$$\mu \leftarrow \bar{x} - \frac{\sigma}{\sqrt{n}} Z.$$

Here we use the symbol '$\leftarrow$' for assignment. The above equation indicates the assignment of (the distribution of) $\bar{x} - \sigma Z/\sqrt{n}$ to (the distribution of) μ. The distribution assigned in this case is $N(\bar{x}, \sigma^2/n)$ [1, section 6.4.7.1]. For the example of $N(\mu, \sigma^2)$ with unknown σ, equation (2) produces

$$\mu \leftarrow \bar{x} - \frac{s}{\sqrt{n}} T,$$

a scaled and shifted Student's t distribution for μ [1, section 6.4.9.2].

In many problems, a pivotal quantity for μ may not be readily available. Asymptotic arguments can be used to obtain approximate pivotal quantities. If, in finite samples, the distribution of D in (3) depends on an unknown parameter ϕ, that is,

$$\Pr[D \leqslant d] = G(d;\ \phi),$$

the simplest method of obtaining an approximate pivotal quantity is to approximate $G(d; \phi)$ by some specified distribution $\hat{G}(w)$ such as $G(d; \hat{\phi})$, where $\hat{\phi}$ is an estimate of ϕ, and proceed to obtain the distribution of μ as though D were in fact pivotal. For illustrative purposes, suppose that we were not aware of the pivotal quantity T in (2) and that we used Z in (1) to obtain a distribution for μ when σ is unknown. Based on the approximate pivotal method, we would obtain

$$\mu \leftarrow \bar{x} - \frac{\hat{\sigma}}{\sqrt{n}} Z,$$

where $\hat{\sigma} = s$ is the estimate of σ. The distribution assigned would be $N(\bar{x}, s^2/n)$, which is, of course, less 'optimal' than the scaled and shifted Student's t distribution derived earlier for this case [6].

Another approach is to start with an approximate pivotal quantity for μ of the form

$$D^* = \frac{\hat{\mu} - \mu}{\hat{\gamma}}, \tag{5}$$

where $\hat{\mu}$ is an estimator of μ and $\hat{\gamma}$ is a scale estimator. If the distribution of D^*, exact or approximate, can be found, then (5) may be inverted to obtain the distribution of μ. If the distribution of D^* is not tractable, the bootstrap method [7] may be used to estimate the distribution of D^*. Here is an outline of using the bootstrap method to estimate the distribution of D^*. We first calculate $\hat{\mu}$, $\hat{\gamma}$, and the parameter estimate $\hat{\boldsymbol{\xi}}$ of the assumed distribution based on the observed samples $x_1, x_2, \ldots, x_n$. We next generate a bootstrap sample $x_1^*, x_2^*, \ldots, x_n^*$ from the assumed distribution with parameter $\hat{\boldsymbol{\xi}}$ and calculate $\hat{\mu}^*$, $\hat{\gamma}^*$ and

$$d^* = \frac{\hat{\mu} - \hat{\mu}^*}{\hat{\gamma}^*}.$$

By repeating this process and generating n_b bootstrap samples, we can then use these n_b values of d^* to estimate the distribution of D^*. The distribution of μ is thus

$$\mu \leftarrow \hat{\mu} - \hat{\gamma} D^*.$$

For the propagation of distributions of input quantities, we need to generate realizations from the distribution of μ, and hence realizations from the distribution of D^*. This can be accomplished by use of a result, generally known as the *probability integral transform* [8, p 202], which states that if X is a random variable with continuous cumulative distribution function (cdf) $F_X(\cdot)$, then $U = F_X(X) \sim \text{uniform}(0, 1)$. Let $G_{D^*}(\cdot)$ be the distribution function of D^*. We have

$$U = G_{D^*}\left(D^*\right)$$

or

$$D^* = G_{D^*}^{-1}(U). \tag{6}$$

Thus, by applying appropriate inverse interpolation to uniform random deviates, realizations of D^* can be obtained.

One way to assess the goodness of using an approximate pivotal quantity D^* to approximate an exact pivotal quantity D is to plot the cdf of the random variable

$$U^* = F_D\left(D^*\right).$$

Recall that if $D^* = D$ then $U^* \sim \text{uniform}(0, 1)$. Thus if D^* is a good approximation of D, the cdf of U^* should be close to the cdf of a uniform (0, 1), which is a 45° line, for all values of nuisance parameters.

We are now ready to apply the pivotal method to some of the commonly used distributions.

3. Examples

In this section we use several examples to illustrate the pivotal method for assigning probability distributions to input quantities.

Example 1. Supposed X_i, $i = 1, \ldots, n$, are distributed as an exponential(μ) distribution with density function given by

$$f(x) = \frac{1}{\mu}\, e^{-x/\mu}\, I_{[0,\infty)}(x),$$

where $I_A(x)$ is an indicator function, that is, $I_A(x) = 1$ if $x \in A$, and 0 otherwise. The distribution is widely used to describe the random recurrence (in time) of an event [9, p 208]. It is known [8, p 237] that $\sum_{i=1}^{n} X_i$ has a gamma(n, μ) distribution with density function

$$f(y) = \frac{1}{\Gamma(n)\mu^n}\, y^{n-1}\, e^{-y/\mu}\, I_{[0,\infty)}(y).$$

Note that $\sum_{i=1}^{n} X_i/\mu$ is distributed as gamma(n, 1), a distribution that is free of unknown parameters. That is, $\sum_{i=1}^{n} X_i/\mu$ is a pivotal quantity for μ. Consequently,

$$\mu \leftarrow \frac{\sum_{i=1}^{n} x_i}{G_n}, \tag{7}$$

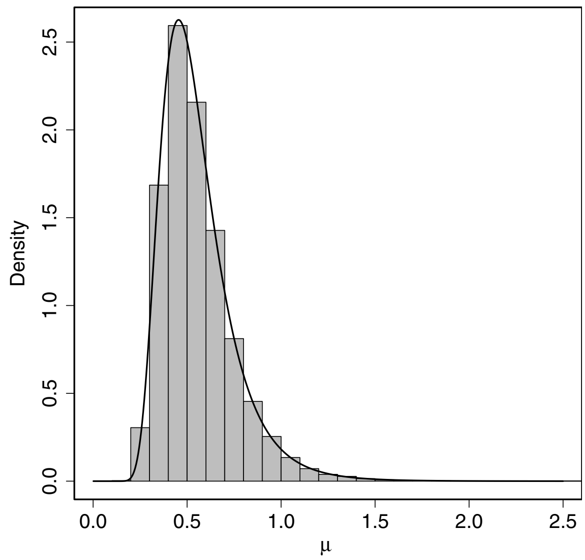

Figure 1. Histogram for 100 000 realizations of μ calculated based on a random sample of size 10 from an exponential(μ) distribution with observed $\sum_{i=1}^{10} x_i = 5$. Superimposed on the histogram is the density.

where $G_n \sim \text{gamma}(n, 1)$. Figure 1 displays the histogram of 100 000 realizations of μ with $n = 10$ and $\sum_{i=1}^{n} x_i = 5$. The realizations were generated by use of the following R program [10].

```
n = 10
sumx = 5
nsample = 100000
mu = sumx/rgamma(nsample, n, 1)
```

Superimposed on the histogram is the density function of μ obtained from the inverse gamma distribution in (7).

Example 2. Let $X_i \sim \text{uniform}(0, \mu)$, $i = 1, \ldots, n$. The maximum likelihood estimator (MLE) of μ is

$$X_{(n)} = \max(X_1, X_2, \ldots, X_n).$$

The probability density function of $X_{(n)}$ is given by [8, p 254]

$$f(x) = \frac{n x^{n-1}}{\mu^n}\, I_{[0,\mu]}(x).$$

Let $W = X_{(n)}/\mu$, then the density function of W is

$$f(w) = n w^{n-1}\, I_{[0,1]}(w),$$

a beta distribution with parameters n and 1. Thus $X_{(n)}/\mu$ is a pivotal quantity for μ and

$$\mu \leftarrow \frac{x_{(n)}}{W}, \tag{8}$$

where $W \sim \text{beta}(n, 1)$. Figure 2 displays the histogram of 100 000 realizations of μ with $n = 10$ and $x_{(n)} = 2$. The realizations were generated by use of the following R program.

```
n = 10
xn = 2
nsample = 100000
mu = xn/rbeta(nsample, n, 1)
```

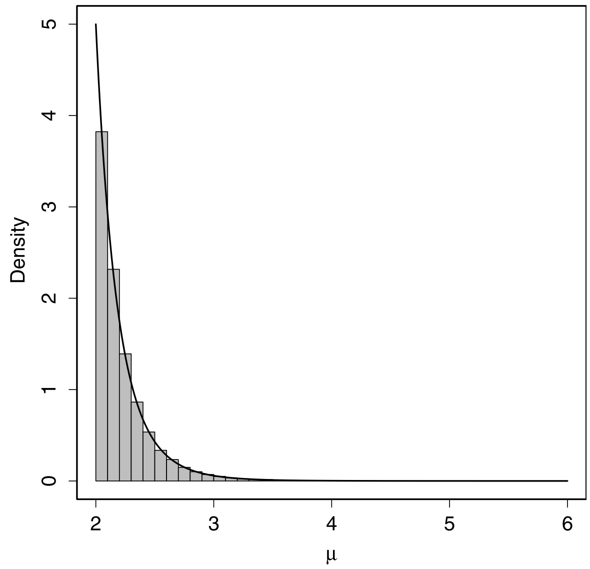

Figure 2. Histogram for 100 000 realizations of μ calculated based on a random sample of size 10 from a uniform$(0,\ \mu)$ distribution with observed $x_{(10)} = 2$. Superimposed on the histogram is the density.

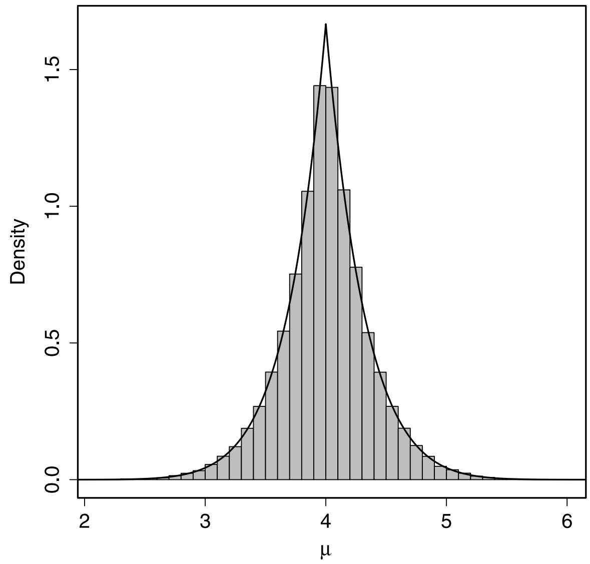

Figure 3. Histogram for 100 000 realizations of μ calculated based on a random sample of size 10 from a uniform$(\mu - 3,\ \mu + 3)$ distribution with observed $(x_{(1)} + x_{(10)})/2 = 4$. Superimposed on the histogram is the density.

Example 3. Let $X_i \sim$ uniform$(\mu - c,\ \mu + c)$, $i = 1, \ldots, n$, where c is a known constant. The midrange $T = (X_{(1)} + X_{(n)})/2$ is a MLE of μ [8, p 282], where $X_{(1)} = \min(X_1,\ X_2,\ \ldots,\ X_n)$. The density function of T is [8, p 256]

$$f(t) = \frac{n}{2c}\left(1 + \frac{t-\mu}{c}\right)^{n-1} I_{[-1,\,0)}\left(\frac{t-\mu}{c}\right) + \frac{n}{2c}\left(1 - \frac{t-\mu}{c}\right)^{n-1} I_{[0,\,1]}\left(\frac{t-\mu}{c}\right).$$

Thus $W = (T - \mu)/c$ is a pivotal quantity for μ and

$$\mu \leftarrow t - c\,W, \tag{9}$$

where W has a density function as

$$f(w) = \frac{n}{2}(1+w)^{n-1}\, I_{[-1,0)}(w) + \frac{n}{2}(1-w)^{n-1}\, I_{[0,1]}(w). \tag{10}$$

The cdf of W is given by

$$F(w) = \begin{cases} (1+w)^n/2 & -1 \leqslant w < 0 \\ 1 - (1-w)^n/2 & 0 \leqslant w \leqslant 1 \\ 1 & 1 < w. \end{cases}$$

To generate a realization of W, we use $W = F^{-1}(U)$, or

$$w = \begin{cases} (2u)^{1/n} - 1 & \text{if } 0 \leqslant u < 0.5 \\ 1 - [2(1-u)]^{1/n} & \text{if } 0.5 \leqslant u \leqslant 1 \end{cases} \tag{11}$$

where u is a random deviate from uniform(0, 1). Figure 3 displays the histogram of 100 000 realizations of μ with $n = 10$, $c = 3$ and $t = 4$. The realizations were generated by use of the following R program.

```
n = 10
c = 3
t = 4
nsample = 100000
U = runif(nsample)
W = U
index = (1:nsample)[U < 0.5]
W[index] = (2 * U[index])^(1/n) - 1
W[-index] = 1 - (2*(1 - U[-index]))^(1/n)
mu = t - c*W
```

Example 4. Let $X_i \sim$ uniform$(\mu - \sqrt{3}\sigma, \mu + \sqrt{3}\sigma)$, $i = 1, \ldots, n$. This uniform distribution has mean μ and variance σ^2. The MLEs of μ and σ are given by [8, p 282]

$$\hat{\mu} = T = \frac{1}{2}\left(X_{(1)} + X_{(n)}\right)$$

$$\hat{\sigma} = S = \frac{1}{2\sqrt{3}}\left(X_{(n)} - X_{(1)}\right).$$

Similar to example 3, the density function of

$$W = \frac{T - \mu}{\sqrt{3}\sigma}$$

is the one given in (10). Thus, W is a pivotal quantity, but it is not a pivotal quantity for μ, because the observed value of W depends on σ, which is unknown. In this case, applying the approximate pivotal method, we replace σ with its MLE $\hat{\sigma}$ and proceed to obtain the probability distribution of μ as though W were in fact pivotal for μ. That is, we obtain

$$\mu \leftarrow t - \sqrt{3}\,s\,W. \tag{12}$$

Realizations of W and hence μ can be generated by use of the equations given in (11). To simplify the presentation, we denote a random variable having the same distribution as the one assigned to μ in (12) by $\widetilde{\mu}_1$; that is, $\widetilde{\mu}_1 = t - \sqrt{3}\,s\,W$.

Next we consider the quantity

$$R = \frac{T - \mu}{\sqrt{3}S}$$

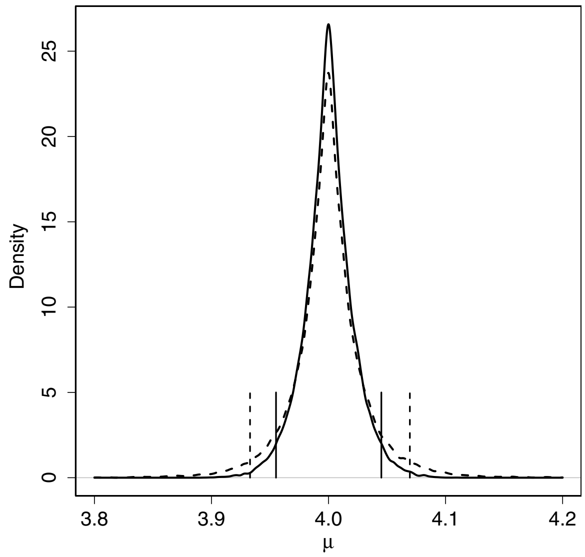

Figure 4. Density plots for $\widetilde{\mu}_1$ (solid) and $\widetilde{\mu}_2$ (dashed) calculated based on a random sample of size 10 from a uniform $(\mu - \sqrt{3}\sigma,\ \mu + \sqrt{3}\sigma)$ distribution with observed $t = 4$ and $s = 0.1$.

and show R is a pivotal quantity for μ. If $X \sim \text{uniform}(\mu - \sqrt{3}\sigma, \mu + \sqrt{3}\sigma)$ then $X = 2\sqrt{3}\sigma U + \mu - \sqrt{3}\sigma$, where $U \sim \text{uniform}(0, 1)$. Consequently, $X_{(1)} = 2\sqrt{3}\sigma U_{(1)} + \mu - \sqrt{3}\sigma$ and $X_{(n)} = 2\sqrt{3}\sigma U_{(n)} + \mu - \sqrt{3}\sigma$, where $U_{(1)} = \min(U_1, U_2, \cdots, U_n)$ and $U_{(n)} = \max(U_1, U_2, \cdots, U_n)$. Using the above results, we have

$$R = \frac{T - \mu}{\sqrt{3}S} = \frac{\left(X_{(1)} + X_{(n)}\right)/2 - \mu}{\left(X_{(n)} - X_{(1)}\right)/2} = \frac{U_{(1)} + U_{(n)} - 1}{U_{(n)} - U_{(1)}},$$

which is free of unknown parameters. Thus R is a pivotal quantity for μ and

$$\mu \leftarrow t - \sqrt{3}\, s\, R. \qquad (13)$$

We denote $\widetilde{\mu}_2 = t - \sqrt{3}\, s\, R$. An R program for generating 50 000 realizations of R with $n = 10$ is listed below.

```
n = 10
nsample = 50000
U = matrix(runif(n*nsample), nrow=nsample)
Umax = apply(U, 1, max)
Umin = apply(U, 1, min)
R = (Umin + Umax - 1)/(Umax - Umin)
```

We now compare the two assignments in (12) and (13). Figure 4 displays density plots of μ for $n = 10$, $t = 4$ and $s = 0.1$. The solid and dashed lines are density estimates based on 50 000 realizations generated by use of (12) and (13), respectively. Figure 4 shows that the density function for $\widetilde{\mu}_1$ (approximate pivotal method) has smaller tails and a more concentrated centre area. Consequently, if $[\widetilde{\mu}_{1,0.025},\ \widetilde{\mu}_{1,0.975}]$ (vertical solid lines) and $[\widetilde{\mu}_{2,0.025},\ \widetilde{\mu}_{2,0.975}]$ (vertical dotted lines) are the 95% confidence intervals for μ based on $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$, respectively, then $[\widetilde{\mu}_{2,0.025},\ \widetilde{\mu}_{2,0.975}]$ will contain $[\widetilde{\mu}_{1,0.025},\ \widetilde{\mu}_{1,0.975}]$ as shown in the plot. Since $\widetilde{\mu}_2$-based intervals are exact, confidence intervals based on $\widetilde{\mu}_1$ will have insufficient probability coverage. We verify this fact by the following simulation study.

Table 1. The actual coverages of nominally 95% confidence intervals for μ.

n	$\widetilde{\mu}_1$	$\widetilde{\mu}_2$
10	0.8743	0.9502
20	0.9155	0.9496
30	0.9284	0.9503
100	0.9436	0.9497

We have

$$\Pr\left[\widetilde{\mu}_{1,0.025} \leqslant \widetilde{\mu}_1 \leqslant \widetilde{\mu}_{1,0.975}\right] = \Pr\left[\frac{t - \widetilde{\mu}_{1,0.975}}{\sqrt{3}s} \leqslant W \leqslant \frac{t - \widetilde{\mu}_{1,0.025}}{\sqrt{3}s}\right].$$

Since the distribution of W does not depend on unknown parameters, we can obtain $w_{0.025}$ and $w_{0.975}$ such that

$$\Pr\left[w_{0.025} \leqslant W \leqslant w_{0.975}\right] = 0.95.$$

Consequently,

$$\frac{t - \widetilde{\mu}_{1,0.975}}{\sqrt{3}s} = w_{0.025}$$

and

$$\frac{t - \widetilde{\mu}_{1,0.025}}{\sqrt{3}s} = w_{0.975}$$

or $\widetilde{\mu}_{1,0.025} = t - \sqrt{3}\, s\, w_{0.975}$ and $\widetilde{\mu}_{1,0.975} = t - \sqrt{3}\, s\, w_{0.025}$. Similarly, the 95% confidence limits for μ based on $\widetilde{\mu}_2$ are $\widetilde{\mu}_{2,0.025} = t - \sqrt{3}\, s\, r_{0.975}$ and $\widetilde{\mu}_{2,0.975} = t - \sqrt{3}\, s\, r_{0.025}$, where

$$\Pr\left[r_{0.025} \leqslant R \leqslant r_{0.975}\right] = 0.95.$$

Using these confidence limits, we obtain the probability coverages of confidence intervals for μ (based on 500 000 Monte Carlo samples) in table 1. The standard error in each entry of table 1, based on the assumption of binomial distribution, is $\sqrt{0.95(1-0.95)/500\,000} = 0.0003$. The values in the table indicate that $\widetilde{\mu}_1$-based intervals have insufficient coverage.

The bootstrap method is not needed, because a pivotal quantity is available for this problem. However, for illustrative purposes, we use the bootstrap method to estimate the distribution of R and obtain the distribution of μ as

$$\mu \leftarrow t - \sqrt{3}\, s\, R^*, \qquad (14)$$

where R^* is the bootstrap estimator of R. We denote $\widetilde{\mu}_3 = t - \sqrt{3}\, s\, R^*$.

The bootstrap procedure, using the observed samples x_i, $i = 1, 2, \ldots, n$, consists of the following steps.

(1) Calculate $t = (x_{(1)} + x_{(n)})/2$ and $s = (x_{(n)} - x_{(1)})/2\sqrt{3}$.

(2) For $i = 1$ to n_b, generate x_j^*, $j = 1, 2, \cdots, n$, from uniform$(t - \sqrt{3}\,s, t + \sqrt{3}\,s)$ and calculate $t^* = (x_{(1)}^* + x_{(n)}^*)/2$, $s^* = (x_{(n)}^* - x_{(1)}^*)/2\sqrt{3}$, and

$$r_i = \frac{t^* - t}{\sqrt{3}\,s^*}.$$

(3) Estimate the distribution of R based on r_i, $i = 1, 2, \ldots, n_b$.

An R program for generating 10 000 r's with $n = 10$, $t = 4$ and $s = 0.1$ is listed below.

```
n = 10
t = 4
s = 0.1
nb = 10000
U = matrix(runif(n*nb, t-sqrt(3)*s,
    t+sqrt(3)*s), nrow=nb)
Umax = apply(U, 1, max)
Umin = apply(U, 1, min)
Bt = (Umin + Umax)/2
Bs = (Umax - Umin)/(2*sqrt(3))
ri = (Bt - t)/(sqrt(3)*Bs)
```

The next step is to use these r_i to construct a cdf for R^* and then use (6) to generate realizations of R^*. Specifically, we construct the empirical cdf of R^*, denoted by $\hat{F}_{R^*}(\cdot)$, based on r_i, $i = 1, 2, \cdots, n_b$, as

$$\hat{F}_{R^*}(r) = \{\text{no. of } r_i \leqslant r\}/n_b.$$

Thus $\hat{F}_{R^*}(r)$ is a 'staircase' function. If n_b is large enough, $\hat{F}_{R^*}(r)$ will be smooth and the inverse interpolation of uniform random deviates can be carried out to obtain realizations of R^*. Suppose

$$\hat{F}_{R^*}(r_0) < u < \hat{F}_{R^*}(r_1);$$

then the value of r such that $\hat{F}_{R^*}(r) = u$, based on an inverse linear interpolation, is given by $r = r_0 + p(r_1 - r_0)$, where

$$p = \frac{u - \hat{F}_{R^*}(r_0)}{\hat{F}_{R^*}(r_1) - \hat{F}_{R^*}(r_0)}.$$

The following R function invInt inversely interpolates uniform random deviates U based on realizations ri.

```
invInt = function(ri, U) {
# inverse linear interpolation to
# uniform random numbers U on the
# cdf of R based on realizations
# ri of R
 ri = sort(ri)
 Fn = ecdf(ri) # empirical cdf F
 Fx = Fn(ri) # values of F(ri)
 # find ix such that
 # Fx[ix] < U < Fx[ix+1]
 ix = findInterval(U, Fx)
 # make sure ix is in [1, length(Fx)-1]
 it = (ix < 1)
 ix[it] = 1
```

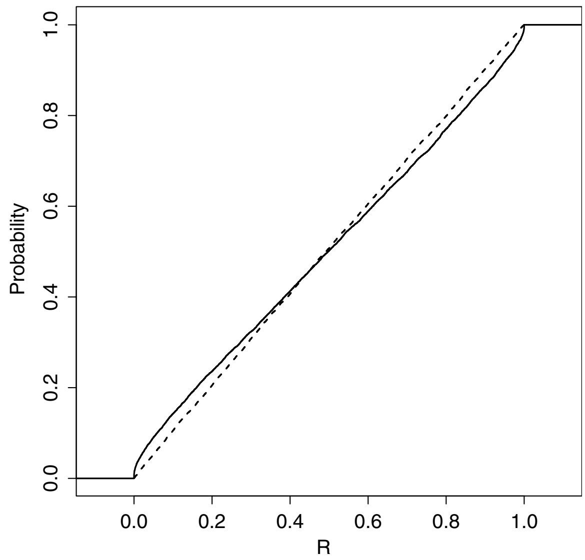

Figure 5. Cdfs for the approximate pivotal (solid) and for the bootstrap (dashed) with $n = 10$.

```
 it = (ix > (length(Fx) - 1))
 ix[it] = length(Fx) - 1
 #
 p = (U - Fx[ix])/(Fx[ix+1] - Fx[ix])
 it = (p < 0)
 p[it] = 0
 out = ri[ix] + p*(ri[ix+1] - ri[ix])
 out
}
```

An R program for generating 50 000 realizations of R^* based on the output from the previous R program is listed below.

```
nsample = 50000
u = runif(nsample)
R = invInt(ri, u)
```

As we mentioned earlier, we can also assess the goodness of approximate pivotal quantities by comparing their cdfs with that of a uniform(0, 1) distribution. Figure 5 displays the cdfs of random variables $F_W((T - \mu)/\sqrt{3}S)$ (solid line) and $F_{R^*}((T - \mu)/\sqrt{3}S)$ (dashed line) for $n = 10$. The dashed line, corresponding to the bootstrap method, appears identical to the cdf of a uniform(0, 1) distribution, which is a 45° line. The solid line, corresponding to the approximate pivotal method, departs from the 45° line at both ends. As n increases, the departures should become smaller (see table 1).

Example 5. Suppose X_i, $i = 1, \ldots, n$, are distributed as a (symmetric) triangular(a, b) distribution with density function given by

$$f(x) = \begin{cases} \dfrac{x - a}{(b - a)^2/4} & a \leqslant x < (a + b)/2 \\ \dfrac{b - x}{(b - a)^2/4} & (a + b)/2 \leqslant x \leqslant b. \end{cases}$$

This distribution is useful in modelling random variables with finite bounds. Let the desired input quantity μ be the mean of the triangular distribution, i.e. $\mu = (a + b)/2$.

Before we develop a pivotal quantity, we observe that if $X \sim \text{triangular}(a, b)$ and $V \sim \text{triangular}(0, 1)$ then $X = (b-a)V + a$. That is, the relationship is identical to the one in the uniform example. Thus, we consider a similar quantity used in the uniform case,

$$W = \frac{T - \mu}{R},$$

for μ, where $T = (X_{(1)} + X_{(n)})/2$ and $R = X_{(n)} - X_{(1)}$. We now show that W is a pivotal quantity for μ. We have

$$W = \frac{T-\mu}{R} = \frac{\left(X_{(1)} + X_{(n)}\right)/2 - \mu}{X_{(n)} - X_{(1)}}$$
$$= \frac{\left(V_{(1)} + V_{(n)} - 1\right)/2}{V_{(n)} - V_{(1)}},$$

where $V_{(1)} = \min(V_1, V_2, \cdots, V_n)$, $V_{(n)} = \max(V_1, V_2, \cdots, V_n)$ and $V_i \sim \text{triangular}(0, 1)$.

We can also use the bootstrap method to estimate the distribution of W and obtain

$$\mu \leftarrow t - r\,W^*, \tag{15}$$

where t and r are realized values of T and R, and W^* is the bootstrap estimator of W. The bootstrap procedure consists of the following steps.

(1) Observe $x_1, \ldots, x_n$ from a triangular(a, b) distribution, calculate $t = (x_{(1)} + x_{(n)})/2$ and $r = x_{(n)} - x_{(1)}$.
(2) For $i = 1$ to n_b, generate x_j^*, $j = 1, 2, \cdots, n$, from triangular$(x_{(1)}, x_{(n)})$ and calculate $t^* = (x_{(1)}^* + x_{(n)}^*)/2$, $r^* = x_{(n)}^* - x_{(1)}$, and

$$w_i = \frac{t^* - t}{r^*}.$$

(3) Estimate the distribution of W based on w_i, $i = 1, 2, \cdots, n_b$.

An `R` program for generating 100 000 realizations of μ based on 50 000 bootstrap samples with $n = 15$, $x_{(1)} = 1.2$ and $x_{(15)} = 4.8$ is listed below.

```
n = 15
xmin = 1.2
xmax = 4.8
tval = (xmin + xmax)/2
rval = xmax - xmin
nb = 50000
uu = matrix(rtri(n*nb, xmin, xmax),
     nrow=nb)
umax = apply(uu, 1, max)
umin = apply(uu, 1, min)
wi = ((umin + umax)/2 - tval)/(umax - umin)
nsample = 100000
u = runif(nsample)
W = invInt(wi, u)
mu = tval - rval*W
```

The histogram of the realizations of μ is shown in figure 6.

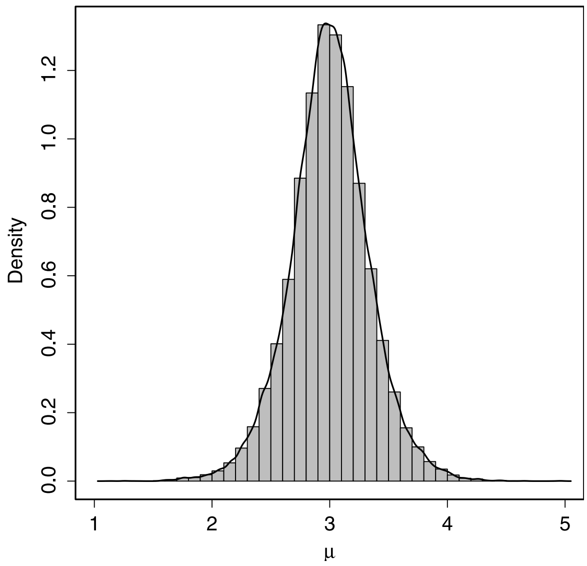

Figure 6. Histogram for 100 000 realizations of μ calculated based on a random sample of size 15 from a triangular distribution with observed $x_{(1)} = 1.2$ and $x_{(15)} = 4.8$. Superimposed on the histogram is the kernel density estimate.

4. Discussion

In this paper we proposed a method for assigning a probability distribution to an input quantity μ when repeated measurements are available for estimating μ. The method is based on an exact or approximate pivotal quantity of μ and is easily applied.

The fiducial method, which also associates a probability distribution with μ, can be considered as a generalization of the pivotal method. In fact, if an exact pivotal quantity of μ exists, the fiducial distribution of μ coincides with the probability distribution of μ obtained with the pivotal method. Since a fiducial recipe is available for arbitrary statistical models [11], the fiducial method provides a general approach for obtaining a probability distribution of μ and is particularly useful in a more complicated situation, for example, see [12]. On the other hand, the pivotal method, with the aid of the bootstrap procedure, provides a simple approach for assigning a probability distribution to μ in commonly used models as illustrated by the examples in section 3. A caution about the bootstrap method is that it requires moderate sample sizes.

We also need to emphasize that exact pivotal quantities are not always easily found. In many applications one has to rely on approximate pivotal quantities. The selection of an approximate pivotal quantity for the problem can be based on asymptotic principle or other considerations. For example, if $T = T(X_1, X_2, \cdots, X_n)$ is the MLE of μ based on a random sample of size n from a distribution $f(x;\, \mu)$, then, under certain regularity conditions, T is asymptotically normally distributed with mean μ and variance

$$\sigma^2(\mu) = \left(n\mathrm{E}\left[\left(\frac{\partial}{\partial\mu}\log f(X;\, \mu)\right)^2\right]\right)^{-1}$$

[8, p 359]. As a result, the following approximate pivotal quantity

$$\frac{T-\mu}{\sigma(T)}$$

can be used to derive a distribution for μ.

Finally, certain optimal properties associated with the pivotal method have not been discussed in this paper. Analogous to the notion of a more 'precise' confidence distribution discussed in [5, 6], we could compare the 'preciseness' of probability distributions obtained with different pivotal quantities (exact or approximate). However, since the confidence distribution is not a probability distribution, a more rigorous treatment of this subject requires further studies.

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