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Least squares generalized inferences in unbalanced two-component normal mixed linear model

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Abstract In this paper, we make use of least squares idea to construct new fiducial generalized pivotal quantities of variance components in two-component normal mixed linear model, then obtain generalized confidence intervals for two variance components and the ratio of the two variance components. The simulation results demonstrate that the new method performs very well in terms of both empirical coverage probability and average interval length. The newly proposed method also is illustrated by a real data example.

Keywords Variance component · Least squares · Fiducial generalized pivotal quantity · Fiducial generalized confidence interval

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1 Introduction

Unbalanced normal mixed linear model with two variance components occupied an important position in the class of random-effects and mixed-effects linear models. This model has proven useful to practitioners in a variety of fields, such as animal breeding studies (Henderson 1984), industrial producing process management (Montgomery 1997), etc. Because variance components are used to account for different sources of variation, it is important to estimate them and some functions of them, such as the ratios of the two variance components. Let σ_α^2 and σ_ε^2 denote variances associated with two sources that influence a response, respectively, then $\eta = \frac{\sigma_\alpha^2}{\sigma_\varepsilon^2}$ and $\rho = \frac{\sigma_\alpha^2}{\sigma_\alpha^2+\sigma_\varepsilon^2}$ measure the importance of one effect compared to the other, these two parameters also are popular to be considered by many researchers.

There are numerous literatures that researched inferences about the above mentioned parameters, such as Olsen et al. (1976), Thomas and Hultquist (1978), Burdick and Graybill (1984), Harville and Fenech (1985), Burdick and Eickman (1986), Fenech and Harville (1991), Burch and Iyer (1997), Hartung and Knapp (2000), Arendacká (2005), Li and Li (2007) and Lidong et al. (2008), etc. Most of these papers researched the parameters of interest by classical exact or asymptotic frequentist methods, while some of recent papers used the idea of generalized inference or fiducial inference. For example, Arendacká (2005) and Li and Li (2007) gave some good generalized inference results based on the generalized inference idea of Weerahandi (1993). Hannig et al. (2006) discussed the connection between generalized inference and fiducial inference and provided a recipe for constructing fiducial intervals. Using the fiducial method described by Hannig et al. (2006), Lidong et al. (2008) constructed a series of fiducial generalized intervals for the aforementioned parameters and their simulation results demonstrated that the proposed fiducial generalized confidence intervals have satisfactory performance in terms of coverage probability, as well as shorter average confidence interval lengths overall.

In this paper, using the least squares idea, a kind of new fiducial generalized confidence intervals for σ_α^2 and ρ are constructed in the framework of unbalanced normal mixed linear model with two variance components. The normal mixed linear model under consideration is

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \boldsymbol{\varepsilon}, \tag{1}$$

where $\mathbf{Y}$ is $N\times 1$ observable random vector, $\boldsymbol{\beta}$ is $p\times 1$ unknown parameters, both $\mathbf{u}$ and $\boldsymbol{\varepsilon}$ are independent unobservable random vectors, the dimension are $a\times 1$ and $N\times 1$, respectively. $\mathbf{X}$ and $\mathbf{Z}$ are known incidence matrixes of sizes $N\times p$ and $N\times a$. Without loss of generality, we assume that $rank(\mathbf{X}) = p$, $\mathbf{u} \sim \mathbf{N}_a(\mathbf{0}, \sigma_\alpha^2\mathbf{A})$, $\boldsymbol{\varepsilon} \sim \mathbf{N}_N(\mathbf{0}, \sigma_\varepsilon^2\mathbf{I}_N)$, where $\mathbf{A}$ is a known matrix that describes the degree to which the elements of the vector $\mathbf{u}$ covary. It is worth noting that the standard unbalanced one-way random model given by

$$Y_{ij} = \mu + u_i + \varepsilon_{ij}, \quad i = 1,\ldots,a;\ j = 1,\ldots,n_i, \tag{2}$$

is a special case of model (1).

The rest of this paper is organized as follows. Section 2 reviews the concept of generalized fiducial method for obtaining confidence intervals. In Sect. 3, some preliminaries about the two-component mixed linear model are introduced at first. The new least square generalized fiducial pivotal quantities are constructed, then these new pivotal quantities are applied to derive generalized fiducial confidence intervals for σ_α^2 and ρ. Section 4 provides details of the simulation studies, along with a discussion of the simulation results. One previously published data example is illustrated in Sect. 5. Section 6 concludes with summary discussions.

2 Review of generalized inference and fiducial generalized inference

The concepts of the generalized test variable and generalized p value were first introduced by Tsui and Weerahandi (1989) to deal with some nontrivial statistical testing problems. These problems involve nuisance parameters in such a fashion that the derivation of a conventional standard pivot is not possible. Weerahandi (1993) proposed generalized pivotal quantity (GPQ) for constructing confidence intervals of the parameter of interest while nuisance parameters existing. These ideas have turned out to be satisfactory in obtaining tests and confidence intervals for many complex problems. In the following years, a lot of applications about the generalized p value and generalized pivotal quantity emerged, such as Zhou and Mathew (1994), Weerahandi (1995, 2004), Krishnamoorthy and Mathew (2003), Roy and Mathew (2005), Tian (2006), Ye and Wang (2009), Li and Li (2007), Liu and Xu (2010), among others. However, there are no published theoretic conclusions discussing either small sample properties or asymptotic behavior of generalized confidence intervals or generalized test until Hannig et al. (2006) pointed out the connection between generalized inference and fiducial inference and furthermore proposed a new concept of fiducial generalized pivotal quantity (FGPQ). Hannig et al. (2006) showed that under some mild conditions, generalized confidence intervals constructed using FGPQs have correct frequentist coverage, at least asymptotically, and proposed three general approaches for constructing FGPQs that most of the previous GPQs can be derived by them. Almost at the same time, Li et al. (2005) and Xu and Li (2006) also found this connection and obtained another general method to construct GPQ.

We listed the definition of FGPQ in Hannig et al. (2006) for convenience of citing later.

Definition 1 Let $\mathbb{S} \in \mathbb{R}^k$ denote an observable random vector whose distribution is indexed by a (possibly vector) parameter $\xi \in \mathbb{R}^p$. Suppose that we are interested in making inferences about $\theta = \pi(\xi) \in \mathbb{R}^q (q \geq 1)$. Let $\mathbb{S}^*$ represent an independent copy of $\mathbb{S}$. We use s and s^* to denote realized values of $\mathbb{S}$ and $\mathbb{S}^*$. A FGPQ for θ , denoted by $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^*, \xi)$ is a function of $(\mathbb{S}, \mathbb{S}^*, \xi)$ with the following properties:

(FGPQ1) The conditional distribution of $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^*, \xi)$, conditional on $\mathbb{S} = s$, is free of ξ.
(FGPQ2) For every allowable $s \in \mathbb{R}^k$, $\mathcal{R}_\theta(s, s, \xi) = \theta$.

The percentiles of $\mathcal{R}_\theta(s, \mathbb{S}^*, \xi)$ are used to construct fiducial generalized confidence intervals for θ. For further details and applications of FGPQ and fiducial generalized inference, we refer readers to Hannig et al. (2006) and Hannig (2009).

3 New least squares fiducial generalized pivotal quantities and confidence intervals

3.1 Some preliminaries of the two-component normal mixed linear model

According to model (1), it is easy to show that

$$\mathbf{Y} \sim \mathbf{N}_N(\mathbf{X}\boldsymbol{\beta}, \sigma_\varepsilon^2\mathbf{I}_N + \sigma_\alpha^2\mathbf{ZAZ}') \tag{3}$$

Let $\mathbf{H}$ be $N \times (N-p)$ matrix such that $\mathbf{HH}' = \mathbf{I}_N - \mathbf{X}(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$ and $\mathbf{H}'\mathbf{H} = \mathbf{I}_{N-p}$, it follows that

$$\mathbf{H}'\mathbf{Y} \sim \mathbf{N}_N(\mathbf{0}, \sigma_\varepsilon^2\mathbf{I}_{N-p} + \sigma_\alpha^2\mathbf{G}), \tag{4}$$

where $\mathbf{G} = \mathbf{H}'\mathbf{ZAZ}'\mathbf{H}$. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_d \geq 0$ be the distinct eigenvalues of $\mathbf{G}$, with multiplicities $r_1, r_2, \ldots, r_d$. Let $\mathbf{P} = [\mathbf{P}_1, \ldots, \mathbf{P}_d]$ be $(N-p) \times (N-p)$ orthogonal matrix such that $\mathbf{P}'\mathbf{GP} = \text{diag}(\lambda_1\mathbf{1}'_{r_1}, \ldots, \lambda_d\mathbf{1}'_{r_d})$, where $\mathbf{P}_i, i = 1, \ldots, d$ corresponding to λ_i is of size $(N-p) \times r_i$. Define

$$V_i = \mathbf{Y}'\mathbf{HP}_i\mathbf{P}_i'\mathbf{H}'\mathbf{Y}, \quad i = 1, \ldots, d. \tag{5}$$

Olsen et al. (1976) proved that $(V_1, \ldots, V_d)$ is minimally sufficient for $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ under (3). Furthermore,

$$U_i = \frac{V_i}{\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2} \sim \chi^2_{r_i}, \quad i = 1, \ldots, d, \tag{6}$$

and $U_i, i = 1, \ldots, d$ are mutually independent, where χ^2_υ represents a central chi-squared distribution with υ degrees of freedom.

3.2 Construction of the new least squares FGPQ

In order to construct the new FGPQ, we start with the minimally sufficient statistics, that is, $(V_1, \ldots, V_d)$. When $d = 2$, there is an invertible relationship between (V_1, V_2) and $(\sigma_\alpha^2, \sigma_\varepsilon^2)$, so it is straightforward to make fiducial inference based on Hannig et al. (2006). Hereinafter, we always assume that $d > 2$. Notice that formulae (6) can be rewritten as follows,

$$\begin{cases} V_1 = (\lambda_1\sigma_\alpha^2 + \sigma_\varepsilon^2)U_1 \\ V_2 = (\lambda_2\sigma_\alpha^2 + \sigma_\varepsilon^2)U_2 \\ \quad\vdots \\ V_d = (\lambda_d\sigma_\alpha^2 + \sigma_\varepsilon^2)U_d \end{cases} \tag{7}$$

As Lidong et al. (2008) noted, (7) provides a structural representation for the observable random vector $\mathbf{V} = (V_1, \ldots, V_d)$ in terms of the random vector $\mathbf{U} = (U_1, \ldots, U_d)$ whose distribution is completely known. $U_i, i = 1, \ldots, d$ are independent, with each U_i having a chi-squared distribution with r_i degrees of freedom. u_i and v_i are denoted as the realized values of U_i and V_i for $i = 1, \ldots, d$, respectively.

When $d > 2$, it is impossible to use the structural method mentioned in Hannig et al. (2006) to construct FGPQs because the number of effective equations is larger than the number of parameter of interest. However, we can use the method of least squares to solve the equation set and construct FGPQs through the least squares solutions.

Define

$$D = \sum_{i=1}^{d} \left[V_i - \left(\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2 \right) U_i \right]^2, \tag{8}$$

the values of $\sigma_\alpha^2, \sigma_\varepsilon^2$ that minimize D can be derived by differentiating (8) with respect to σ_α^2 and σ_ε^2, then set these partial derivatives equal to zero, we obtain:

$$\begin{cases} \sigma_\alpha^2 \sum_{i=1}^{d} \lambda_i^2 U_i^2 + \sigma_\varepsilon^2 \sum_{i=1}^{d} \lambda_i U_i^2 = \sum_{i=1}^{d} V_i \lambda_i U_i \\ \sigma_\alpha^2 \sum_{i=1}^{d} \lambda_i U_i^2 + \sigma_\varepsilon^2 \sum_{i=1}^{d} U_i^2 = \sum_{i=1}^{d} V_i U_i \end{cases}. \tag{9}$$

Denote

$$\begin{aligned} d_0 &= \sum_{i=1}^{d} U_i^2 \sum_{i=1}^{d} \lambda_i^2 U_i^2 - \left(\sum_{i=1}^{d} \lambda_i U_i^2 \right)^2, \\ d_1 &= \sum_{i=1}^{d} U_i^2 \sum_{i=1}^{d} \lambda_i V_i U_i - \sum_{i=1}^{d} \lambda_i U_i^2 \sum_{i=1}^{d} V_i U_i, \\ d_2 &= \sum_{i=1}^{d} \lambda_i^2 U_i^2 \sum_{i=1}^{d} V_i U_i - \sum_{i=1}^{d} \lambda_i U_i^2 \sum_{i=1}^{d} \lambda_i V_i U_i, \end{aligned} \tag{10}$$

then the least squares solutions of σ_α^2 and σ_ε^2 are given by

$$\sigma_\alpha^2 = \frac{d_1}{d_0}, \quad \sigma_\varepsilon^2 = \frac{d_2}{d_0}. \tag{11}$$

According to Hannig et al. (2006) and Xu and Li (2006), FGPQs of $\sigma_\alpha^2, \sigma_\varepsilon^2$ can be constructed through (11).

Let

$$\begin{aligned} d_0^* &= \sum_{i=1}^{d} U_i^{*2} \sum_{i=1}^{d} \lambda_i^2 U_i^{*2} - \left(\sum_{i=1}^{d} \lambda_i U_i^{*2} \right)^2 \\ &= \sum_{i=1}^{d} \left(\frac{V_i^*}{\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2} \right)^2 \sum_{i=1}^{d} \lambda_i^2 \left(\frac{V_i^*}{\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2} \right)^2 - \left(\sum_{i=1}^{d} \lambda_i \left(\frac{V_i^*}{\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2} \right)^2 \right)^2, \\ d_1^* &= \sum_{i=1}^{d} U_i^{*2} \sum_{i=1}^{d} \lambda_i V_i U_i^* - \sum_{i=1}^{d} \lambda_i U_i^{*2} \sum_{i=1}^{d} V_i U_i^* \end{aligned}$$

$$= \sum_{i=1}^{d}\left(\frac{V_i^*}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2}\right)^2 \sum_{i=1}^{d}\frac{\lambda_i V_i V_i^*}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2} - \sum_{i=1}^{d}\lambda_i\left(\frac{V_i^*}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2}\right)^2 \sum_{i=1}^{d}\frac{V_i V_i^*}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2},$$

$$d_2^* = \sum_{i=1}^{d}\lambda_i^2 U_i^{*2}\sum_{i=1}^{d}V_i U_i^* - \sum_{i=1}^{d}\lambda_i U_i^{*2}\sum_{i=1}^{d}\lambda_i V_i U_i^*$$

$$= \sum_{i=1}^{d}\left(\frac{\lambda_i V_i^*}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2}\right)^2 \sum_{i=1}^{d}\frac{V_i V_i^*}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2} - \sum_{i=1}^{d}\lambda_i\left(\frac{V_i^*}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2}\right)^2 \sum_{i=1}^{d}\frac{\lambda_i V_i V_i^*}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2}, \tag{12}$$

where λ_i, r_i are known, U_i^* and V_i^* are the independent copies of U_i and V_i, respectively, and V_i is observable. Define

$$\begin{cases} \mathcal{R}_{\sigma_\alpha^2} = \frac{d_1^*}{d_0^*} \\ \mathcal{R}_{\sigma_\varepsilon^2} = \frac{d_2^*}{d_0^*} \end{cases} \tag{13}$$

Theorem 1 $\mathcal{R}_{\sigma_\alpha^2}, \mathcal{R}_{\sigma_\varepsilon^2}$ *defined by* (13) *are FGPQs of* $\sigma_\alpha^2, \sigma_\varepsilon^2$, *respectively.*

Proof Notice that

$$U_i^* = \frac{V_i^*}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2} \sim \chi_{r_i}^2, \quad i = 1, 2, \ldots, d,$$

provided $\mathbf{V} = \mathbf{v}$, where $\mathbf{v}$ is the observed value of $\mathbf{V}$, the definitions of d_0^*, d_1^*, d_2^* show that distributions of $\mathcal{R}_{\sigma_\alpha^2}, \mathcal{R}_{\sigma_\varepsilon^2}$ are independent of any unknown parameters. When $V_i^* = V_i$ for $i = 1, 2, \ldots, d$, it is readily verified that $\mathcal{R}_{\sigma_\alpha^2} = \sigma_\alpha^2, \mathcal{R}_{\sigma_\varepsilon^2} = \sigma_\varepsilon^2$, so both of the requirements of Definition 1 are satisfied. □

In general, it is difficult to calculate the distributions of $\mathcal{R}_{\sigma_\alpha^2}, \mathcal{R}_{\sigma_\varepsilon^2}$, Monte Carlo method is useful in practice calculation. Next we make use of these new least squares FGPQs (LSFGPQs) to construct new confidence intervals for the parameters of interest in the two-component normal mixed linear model.

3.3 Least squares fiducial generalized confidence intervals (LSFGCIs) for σ_α^2 and ρ

3.3.1 LSFGCI for σ_α^2

A LSFGCI for σ_α^2 can be easily derived from the LSFGPQ $\mathcal{R}_{\sigma_\alpha^2} = \frac{d_1^*}{d_0^*}$. Let $l_\alpha(\mathbf{v}), l_\beta(\mathbf{v})$ be the $\alpha, \beta(\alpha < \beta)$ quantiles of $\mathcal{R}_{\sigma_\alpha^2}$ respectively, given $\mathbf{V} = \mathbf{v}$, then a two-sided $100(\beta - \alpha)\%$ LSFGCI for σ_α^2 is derived by

$$\left[\max\left(0, l_\alpha(\mathbf{v})\right), \max\left(0, l_\beta(\mathbf{v})\right)\right]. \tag{14}$$

The LSFGCI for σ_α^2 can be computed by Monte Carlo simulation, the following is the algorithm.

Algorithm 1 Step 1: Compute $\lambda_i, r_i, V_i, i = 1, \ldots, d$ for given values $\mathbf{Y}, \mathbf{X}, \mathbf{Z}$.
Step 2: Generate random numbers from $\chi^2_{r_i}$ as the realized values of U_i for $i = 1, \ldots, d$.
Step 3: Compute d_0^*, d_1^* and d_1^*/d_0^* according to (12).
Step 4: Repeat Step 2 to 3 for L times, and sort the derived d_1^*/d_0^*.
Step 5: Find the $\frac{\alpha}{2}$ and $1 - \frac{\alpha}{2}$ empirical quantiles from Step 4, then the LSFGCI is obtained by using (14).

3.3.2 LSFGCI for ρ

In order to construct LSFGCI for ρ, we construct the LSFGPQ for ρ at first. Similar as the proof of Theorem 1, it is straightforward to show that

$$\mathcal{R}_\rho = \frac{\mathcal{R}_{\sigma_\alpha^2}}{\mathcal{R}_{\sigma_\alpha^2} + \mathcal{R}_{\sigma_\varepsilon^2}} = \frac{d_1^*}{d_1^* + d_2^*} \tag{15}$$

is a FGPQ for ρ. Hence, we can easily get the LSFGCI of ρ just like the procedure in last subsection. Let $l_\alpha^\rho(\mathbf{v}), l_\beta^\rho(\mathbf{v})$ be the $\alpha, \beta(\alpha < \beta)$ quantiles of $\mathcal{R}_\rho$ respectively, given $\mathbf{V} = \mathbf{v}$, then a two-sided $100(\beta - \alpha)$ % LSFGCI for ρ is derived by

$$\left[\max\left(0, l_\alpha^\rho(\mathbf{v})\right), \ \min\left(1, \max\left(0, l_\beta^\rho(\mathbf{v})\right)\right)\right]. \tag{16}$$

The algorithm for obtaining the LSFGCI of ρ is also similar as the one for σ_α^2.

Algorithm 2 Step 1: Compute $\lambda_i, r_i, V_i, i = 1, \ldots, d$ for given values $\mathbf{Y}, \mathbf{X}, \mathbf{Z}$.
Step 2: Generate random numbers from $\chi^2_{r_i}$ as the realized values of U_i for $i = 1, \ldots, d$.
Step 3: Compute d_1^*, d_2^* and $\frac{d_1^*}{d_1^*+d_2^*}$ according to (12).
Step 4: Repeat Step 2 to 3 for L times, and sort the derived $\frac{d_1^*}{d_1^*+d_2^*}$.
Step 5: Find the $\frac{\alpha}{2}$ and $1 - \frac{\alpha}{2}$ empirical quantiles from Step 4, then the LSFGCI is obtained by using (16).

4 Simulation studies

In this section, we compare the performance of the new approach with several previously proposed methods. The coverage probability of a confidence interval on σ_α^2 depends on the design (e.g., number of within group measurements, $n_1, \ldots, n_a$) as well as on the values of $\eta = \frac{\sigma_\alpha^2}{\sigma_\varepsilon^2}$. The degree of imbalance of the design in the case of a one-way random-effects model has been quantified by the measure Φ in Ahrens and Pincus (1981), defined as $\Phi = a\widetilde{n}/N$ with $N = \sum_{i=1}^a n_i, \widetilde{n} = a/(\sum_{i=1}^a (1/n_i))$, actually, Φ is the ratio of the harmonic mean to the arithmetic mean of $(n_1, \ldots, n_a)$,

Table 1 Simulation designs used in the simulation study

Design	Φ	a	n_i					
1	0.068	6	1	1	1	1	1	100
2	0.130	6	2	2	2	2	2	100
3	0.187	3	2	5	60			
4	0.265	5	1	1	10	10	20	
5	0.357	3	1	1	10			
6	0.410	5	4	4	4	8	48	
7	0.556	4	4	4	20	20		
8	0.700	6	5	10	15	20	25	30
9	0.807	4	2	2	4	6		
10	0.957	6	6	6	8	8	10	10

and it is well known that the harmonic mean is no larger than the arithmetic mean, so there is $0 < \Phi \leq 1$. Note that $\Phi = 1$ if and only if n_is are all equal. For our simulation study, we select ten different unbalanced patterns, the values of Φ range from 0.068 to 0.957 as shown in Table 1. The values selected for $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ are (0.1, 10), (0.5, 10), (1, 10), (0.5, 2), (1, 1), (2, 0.5), (5, 0.2), (10, 0.1), We choose these settings for our study to better investigate the performance of confidence intervals from small to large values of the ratio $\eta = \frac{\sigma_\alpha^2}{\sigma_\varepsilon^2}$. The simulation study is programmed in Matlab. For each setting of sample size n_i and values of $(\sigma_\alpha^2, \sigma_\varepsilon^2)$, we generate 5000 independent data sets and compute two-sided 95 % confidence intervals for σ_α^2 and ρ for each method.

For the inference on σ_α^2, we compare the methods proposed by Burdick and Eickman (1986), Burdick and Graybill (1984), Lidong et al. (2008) with our new least squares fiducial generalized approach, these methods are denoted as BE, BG, EHI and LSF, respectively. The criteria for judging the performance of the methods are the empirical coverage probability and the average length of the confidence interval.

For better understanding the simulation results, the numbers of our simulation studies are graphically summarized in the boxplots of Figs. 1, 2, 3, 4. (For the detailed numerical simulation results, see supplementary materials or contact with the corresponding author). Figures 1 and 2 show the empirical coverage probabilities for settings with ratio $\eta = \frac{\sigma_\alpha^2}{\sigma_\varepsilon^2} < 1$ and for settings with $\eta \geq 1$. Figures 3 and 4 show the differences of the average confidence interval lengths, relative to the new least squares fiducial interval, for all competing procedures for settings with $\eta < 1$ and settings with $\eta \geq 1$. These relative lengths are denoted by RL, which is defined as $RL = \frac{AL_M - AL_{LSF}}{AL_{LSF}}$, where AL_M denotes the average length of a competing interval and AL_{LSF} denotes the average length of the LSF interval. Apparently, the larger the value of RL is than 0, the longer is the average length of interval of M method than that of LSF's interval.

Figures 1 and 2 show that BG procedure is liberal when the ratio η is large. BE and EHI are conservative when the ratio η is <1. These findings agree with the results of Burdick and Eickman (1986) and Lidong et al. (2008). Our new procedure performs very well in both $\eta < 1$ and $\eta \geq 1$ situations.

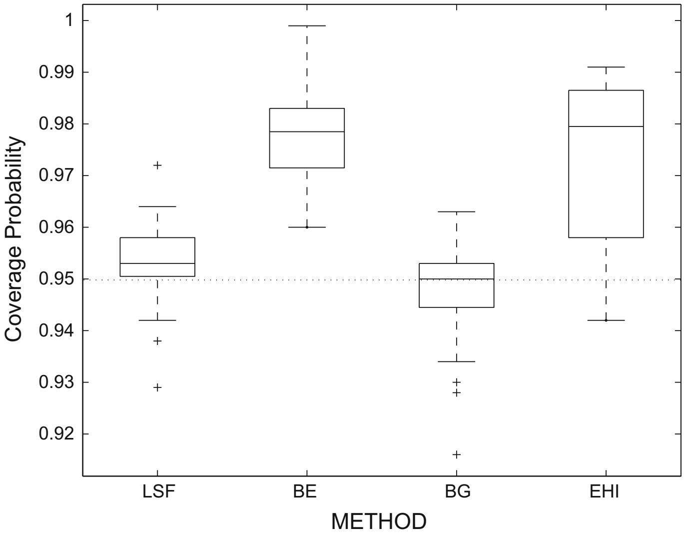

Fig. 1 Empirical coverage probabilities of CI for σ_α^2 for settings with $\eta < 1$

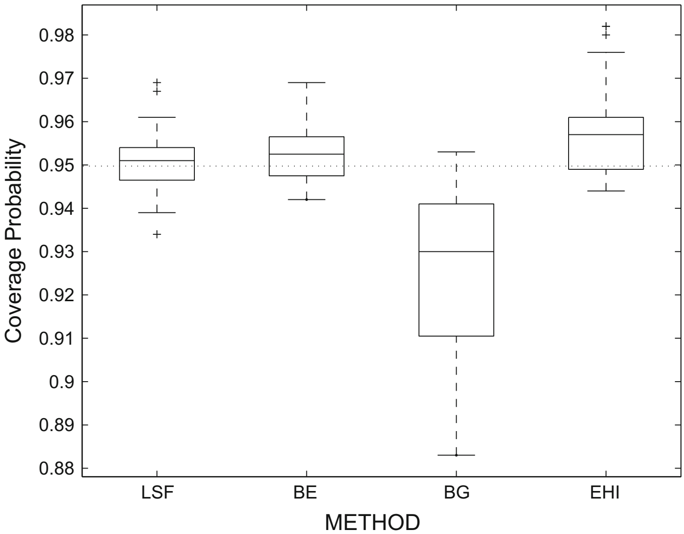

Fig. 2 Empirical coverage probabilities of CI for σ_α^2 for settings with $\eta \geq 1$

Regarding average interval length, according to Figs. 3 and 4, we see that all of the three competing intervals have longer mean of average interval lengths than the new LSF intervals. EHI is better than the other two methods, which also agrees with the results of Lidong et al. (2008). We notice that the LSF method performs better under the situation of $\eta < 1$ than $\eta \geq 1$ according to Figs. 3 and 4, and the average lengths of LSF intervals are much shorter than the average lengths of all other intervals in terms

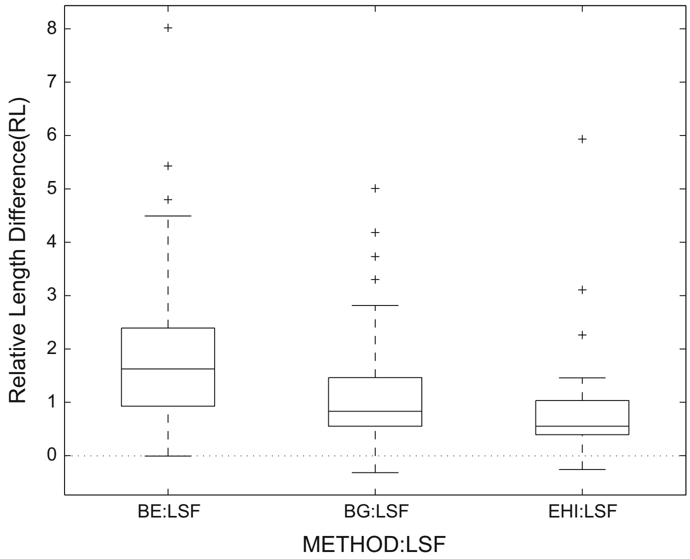

Fig. 3 Relative difference of the average confidence interval length (RL) of CI for σ_α^2 for settings with $\eta < 1$

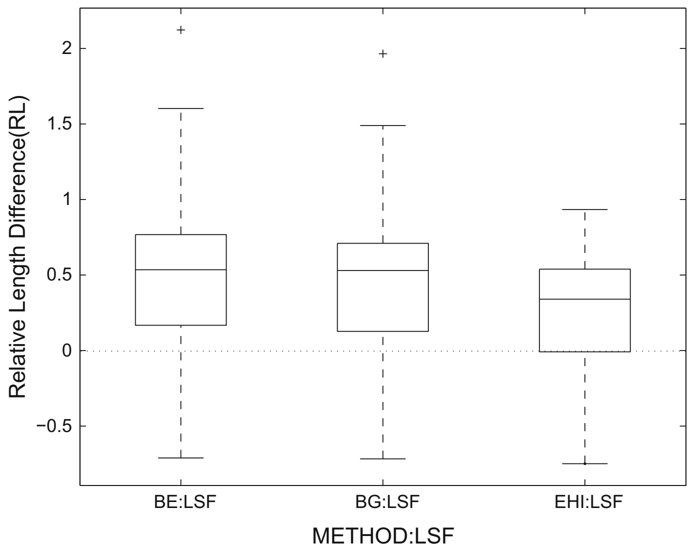

Fig. 4 Relative difference of the average confidence interval length (RL) of CI for σ_α^2 for settings with $\eta \geq 1$

of the mean line of the boxplots for both of the situations. Based on these results, we recommend the LSF intervals for σ_α^2 as a suitable choice for practical use.

In order to investigate the new confidence interval for ρ, we compare the methods proposed by Burch and Iyer (1997), Thomas and Hultquist (1978), Hartung and Knapp (2000) and Lidong et al. (2008) with our new approach, these methods are denoted as

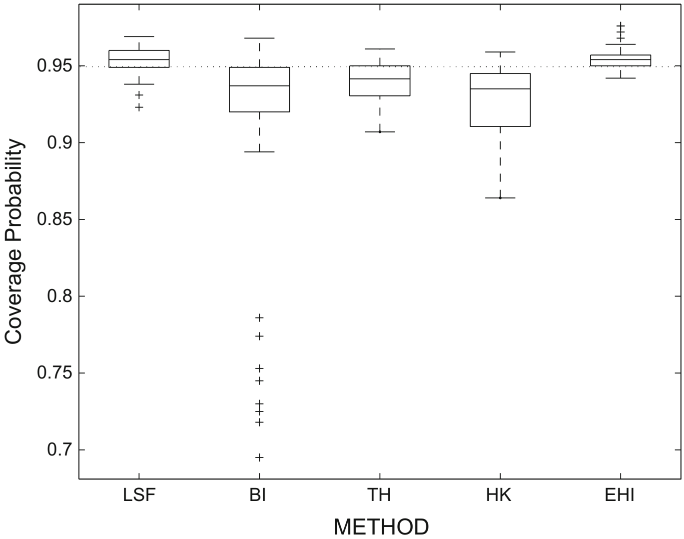

Fig. 5 Empirical coverage probabilities of CI for ρ for settings with $\eta < 1$

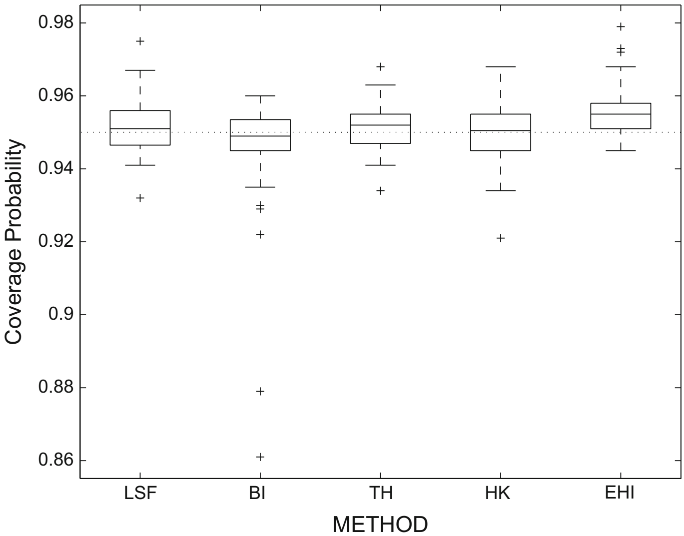

Fig. 6 Empirical coverage probabilities of CI for ρ for settings with $\eta \geq 1$

BI, TH, HK, EHI and LSF, respectively. Figures 5, 6, 7, 8, the graphically summarized boxplots for the simulation results of ρ, are parallel to Figs. 1, 2, 3, 4. Figures 5 and 6 show that the new method also has good performance in terms of coverage probability under the situations of both $\eta < 1$ and $\eta \geq 1$. The coverage probabilities of BI method are too much smaller than the nominal level in some settings, such as $n = (1, 1, 10, 10, 20)$ and $(1, 1, 10)$ with $\eta < 1$. Most of the coverage probabilities of HK method are also smaller than the nominal level when $\eta < 1$, and EHI method is

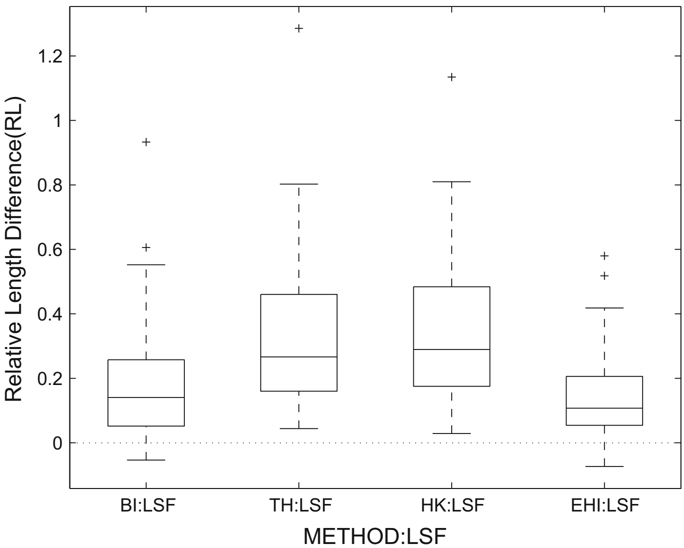

Fig. 7 Relative difference of the average confidence interval length (RL) of CI for ρ for settings with $\eta < 1$

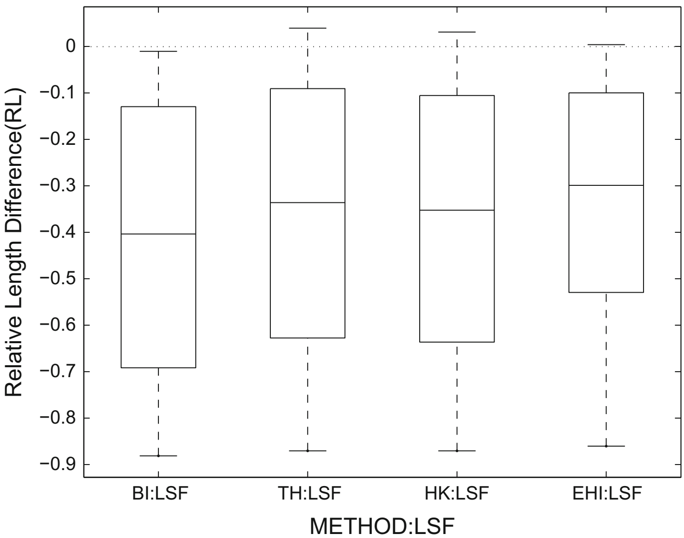

Fig. 8 Relative difference of the average confidence interval length (RL) of CI for ρ for settings with $\eta \geq 1$

conservative as before. Figure 7 shows that the average lengths of LSF intervals are also shorter than the average lengths of all other intervals in terms of the median line of the boxplots. Regarding the average length, EHI method is the second shortest, and BI method shows better results than the other two competing methods, i.e., TH and HK. In addition, the interval lengths of the LSF method is not as good as before according to Fig. 8 under the situation of $\eta \geq 1$.

It is worth noting that the performance of LSF method shows no obvious relation to the value of Φ according to the simulation results of both σ_α^2 and ρ. According to the above discussion, we recommend our new fiducial method while estimating σ_α^2 in unbalanced normal mixed linear models with two variance components. For the inference on the parameter ρ, we also suggest our new method especially for the situation of $\eta < 1$.

5 An example: sire model

For purposes of illustration, we consider the data set used in Harville and Fenech (1985) and Burch (1996). The original data are listed in Table 1 of Harville and Fenech (1985). This data set came from five distinct population lines which include two control lines and three selection lines. Each lamb was progeny of one of 23 rams, and each lamb had a different dam. Age of each dam was recorded as belonging to one of three categories: 1–2, 2–3 years and over 3 years.

Harville and Fenech (1985) considered a unbalanced mixed linear model for this data set as follows:

$$Y_{ijkl} = \mu + \delta_i + \pi_j + s_{jk} + e_{ijkl}, \tag{17}$$

where Y_{ijkl} represent the lth lamb of the kth ram in the jth population line from a dam belonging to the ith age category, $i = 1, 2, 3$; $j = 1, \ldots, 5$. In model (17), $(\delta_1, \delta_2, \delta_3)$ denote the age effects and $(\pi_1, \ldots, \pi_5)$ denote the line effects, these two are fixed effects. The sire (within line) effects $(s_{11}, s_{12}, \ldots, s_{58})$ are random effects that are distributed independently as $N(0, \sigma_\alpha^2)$ and the random errors $(e_{1111}, e_{1112}, \cdots, e_{3582})$ are distributed as $N(0, \sigma_\varepsilon^2)$ independently of each other and of the sire effects. Define $\boldsymbol{\beta} = (\mu, \delta_1, \delta_2, \delta_3, \pi_1, \cdots, \pi_5)'$, $\boldsymbol{\varepsilon} = (e_{1111}, e_{1112}, \cdots, e_{3582})'$ and $\mathbf{u} = (s_{11}, s_{12}, \cdots, s_{58})'$, clearly, model (17) is expressible as a special case of mixed linear model (1). Under model (17), Harville and Fenech (1985) discussed the problem of testing $H_0 : \sigma_\alpha^2 = 0$. Here we consider the confidence interval of σ_α^2 and ρ which also was interpretable as a heritability in biology.

Assume that relation matrix among the rams is $\mathbf{A} = \mathbf{I}_{23}$, that is, the 23 rams are assumed to be unrelated. The number of different eigenvalues of $\mathbf{G} = \mathbf{H}'\mathbf{Z}\mathbf{A}\mathbf{Z}'\mathbf{H}$ is $d = 18$. The eigenvalues range magnitude from $\lambda_1 = 5.087479$ to $\lambda_{18} = 0$, where $\lambda_8 = 2.0$ with multiplicity $r_8 = 2$, $\lambda_{18} = 0$ with multiplicity $r_{18} = 37$, and all the remaining eigenvalues have a multiplicity of one. The method of moments (MOM) estimates and restrictive maximum likelihood (REML) estimates of parameters of interest about this data set are shown in Table 5.

We obtain the new confidence intervals using our proposed approach and compare them with the intervals derived by several existing methods. According to Table 2,

Table 2 Estimates of $\sigma_\alpha^2, \sigma_\varepsilon^2, \rho$

Method	$\widehat{\sigma_\alpha^2}$	$\widehat{\sigma_\varepsilon^2}$	$\widehat{\rho}$
MOM	0.7676	2.7631	0.2174
REML	0.5171	2.9616	0.1486

Table 3 Nominally 95 % confidence intervals on σ_α^2 for the lamb birth-weight data

Method	LSF	EHI
95 % CI	(0, 1.058)	(0, 2.150)

Table 4 Nominally 95 % confidence intervals on ρ for the lamb birth-weight data

Method	LSF	BI	EHI
95 % CI	(0, 0.321)	(0, 0.644)	(0, 0.512)

Table 5 Nominally 95 % confidence intervals on σ_ε^2 for the lamb birth-weight data

Method	LSF	EX	EHI
95 % CI	(1.827, 4.665)	(1.842, 4.638)	(1.996, 5.023)

the MOM and REML estimates of $\eta = \frac{\sigma_\alpha^2}{\sigma_\varepsilon^2}$ are 0.2778 and 0.1746, respectively, both are far <1, so the new intervals should be better than other competing methods based on the simulation conclusions of last section, and the outcomes indeed support our conjecture.

Table 3 demonstrates the LSF and EHI confidence intervals for σ_α^2 and Table 4 presents LSF, BI and EHI confidence intervals for the ρ with 95 % nominal confidence coefficients, respectively. Since the BE, BG, HK and TH methods are proposed under the one-way random effects model, they cannot be used here for this example. Tables 3 and 4 show that the LSF method gives shorter confidence intervals for this data set, and it is in accordance with the simulation results in Sect. 4.

For sake of completeness, we also show the LSF, EHI and the exact confidence intervals(denoted as EX) for σ_ε^2 with 95 % nominal confidence coefficient in Table 5. According to Subsect. 3.3, a LSFGCI for σ_ε^2 can be easily obtained from the LSFGPQ $\mathcal{R}_{\sigma_\varepsilon^2} = \frac{d_2^*}{d_0^*}$. Let $l_\alpha(\mathbf{v}), l_\beta(\mathbf{v})$ be the $\alpha, \beta(\alpha < \beta)$ quantiles of $\mathcal{R}_{\sigma_\varepsilon^2}$ respectively, given $\mathbf{V} = \mathbf{v}$, then a two-sided $100(\beta - \alpha)$ % LSFGCI for σ_ε^2 is given by

$$\left[\max\left(0, l_\alpha\left(\mathbf{v}\right)\right), \max\left(0, l_\beta\left(\mathbf{v}\right)\right)\right]. \tag{18}$$

The LSFGCI for σ_ε^2 can be computed by Monte Carlo simulation, and the algorithm is just parallel to Algorithm 1 or 2.

For the exact confidence interval of σ_ε^2, note that a pure error estimate of σ_ε^2 is given by $\frac{V_d}{r_d}$ if λ_d is equal to 0 in formulae (6), then an exact $100(1-\alpha)$ % confidence interval for σ_ε^2 exists and is given by $\left[\frac{V_d}{\chi^2_{1-\alpha/2}(r_d)}, \frac{V_d}{\chi^2_{\alpha/2}(r_d)}\right]$. Table 5 shows that LSF confidence interval is very close to the exact confidence interval and is shorter than the EHI confidence interval.

In a word, according to the above results, the new LSF procedure performs better than other methods for this lamb birth-weight data set.

6 Concluding remarks

In this paper, we propose a new procedure to construct confidence intervals for σ_α^2, σ_ε^2 and ρ in the two-component normal mixed effects linear model using the least squares idea mingling with fiducial generalized pivotal quantity. Simulation studies are carried out to compare the proposed confidence intervals with several other confidence intervals from the existing literature, the results show that the new procedure is very satisfactory in terms of coverage probability and average interval length. We also use a real data set to illustrate the use of the proposed procedure. These results confirm that the LSFGCIs can be recommended for practical use. It is worth noting that although we focus on confidence interval estimation in this paper, our results can be used to carry out hypothesis tests about the variance components and the related functions of them.

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