Fiducial Inference and Generalizations*

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1 Introduction

The origin of Generalized Fiducial Inference can be traced back to R. A. Fisher (Fisher, 1930, 1933, 1935) who introduced the concept of a fiducial distribution for a parameter, and proposed the use of this fiducial distribution, in place of the Bayesian posterior distribution, for interval estimation of this parameter. In the case of a one-parameter family of distributions, Fisher gave the following definition for a fiducial density $r(\theta)$ of the parameter based on a single observation x for the case where the cdf $F(x, \theta)$ is a monotonic decreasing function of θ:

$$r(\theta) = -\frac{\partial F(x, \theta)}{\partial \theta}. \quad (1)$$

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In simple situations, especially in one parameter families of distributions, Fisher's fiducial intervals turned out to coincide with classical confidence intervals. For multiparameter families of distributions, the fiducial approach led to confidence sets whose frequentist coverage probabilities were close to the claimed confidence levels but they were not exact in the frequentist sense. Fisher's proposal led to major discussions among the prominent statisticians of the 1930's, 40's and 50's (e.g., Dempster, 1966, 1968; Fraser, 1961a,b, 1966, 1968; Jeffreys, 1940; Lindley, 1958; Stevens, 1950). Many of these discussions focused on the nonexactness of the confidence sets and also nonuniqueness of fiducial distributions. The latter part of the 20th century has seen only a handful of publications Barnard (1995); Dawid and Stone (1982); Dawid *et al.* (1973); Salome (1998); Wilkinson (1977) as the fiducial approach fell into disfavor and became a topic of historical interest only.

Recently, the work of Tsui and Weerahandi (1989, 1991) and Weerahandi (1993, 1994, 1995) on generalized confidence intervals and the work of Chiang (2001) on the *surrogate variable method* for obtaining confidence intervals for variance components, led to the realization that there was a connection between these new procedures and fiducial inference. This realization evolved through a series of works (Hannig, 2009b; Hannig *et al.*, 2006b; Iyer and Patterson, 2002; Iyer *et al.*, 2004; Patterson *et al.*, 2004). The strengths and limitations of the fiducial approach is becoming to be better understood, see, especially, Hannig (2009b). In particular, the asymptotic exactness of fiducial confidence sets, under fairly general conditions, was established in Hannig *et al.* (2006b); Hannig (2009a,b).

Subsequently Hannig *et al.* (2003); Iyer *et al.* (2004); McNally *et al.* (2003); Wang and Iyer (2005, 2006a,b) applied this fiducial approach to derive confidence procedures in many important practical problems. Hannig (2009b) extended the initial ideas and proposed a Generalized Fiducial Inference procedure that could be applied to arbitrary classes of models, both parametric and nonparametric, both continuous and discrete. These applications include Bioequivalence (Hannig *et al.*, 2006a), Variance Components (E *et al.*, 2008), Problems of Metrology (Hannig *et al.*, 2007, 2003; Wang and Iyer, 2005, 2006a,b), Interlaboratory Experiments and International Key Comparison Experiments (Iyer *et al.*, 2004), Maximum Mean of a Multivariate Normal Distribution (Wandler and Hannig, 2009), Mixture of a Normal and Cauchy (Glagovskiy, 2006), Wavelet Regression (Hannig and Lee, 2009), Logistic Regression and LD_{50} (E *et al.*, 2009). Recently, other authors have also contributed to research on fiducial methods and related topics (e.g., Berger and Sun, 2008; Wang, 2000; Xu and Li, 2006).

2 Generalized Fiducial Distribution

The idea underlying Generalized Fiducial Inference comes from an extended application of Fisher's fiducial argument, which is briefly described as follows. Generalized Fiducial Inference begins with expressing the relationship between the data, $\mathbf{X}$, and the parameters, $\boldsymbol{\theta}$, as

$$\mathbf{X} = G(\boldsymbol{\theta}, \mathbf{U}), \tag{2}$$

where $G(\cdot, \cdot)$ is termed structural equation, and $\mathbf{U}$ is the random component of the structural equation whose distribution is completely known. The data $\mathbf{X}$ are assumed to be created by generating a random variable $\mathbf{U}$ and plugging it into the structural equation (2).

For simplicity, this subsection only considers the case where the structural relation (2) can be inverted and the inverse $G^{-1}(\cdot, \cdot)$ always exists. Thus, for any observed $\mathbf{x}$ and for any arbitrary $\mathbf{u}$, $\boldsymbol{\theta}$ is obtained as $\boldsymbol{\theta} = G^{-1}(\mathbf{x}, \mathbf{u})$. Fisher's *Fiducial Argument* leads one to define the fiducial distribution for $\boldsymbol{\theta}$ as the distribution of $G^{-1}(\mathbf{x}, \mathbf{U}^\star)$ where $\mathbf{U}^\star$ is an independent copy of $\mathbf{U}$. Equivalently, a sample from the fiducial distribution of $\boldsymbol{\theta}$ can be obtained by generating $\mathbf{U}_i^\star,\ i = 1, \ldots, N$ and using $\boldsymbol{\theta}_i = G^{-1}(\mathbf{x}, \mathbf{U}_i^\star)$. Estimates and confidence intervals for $\boldsymbol{\theta}$ can be obtained based on this sample.

Hannig (2009b) has generalized this to situations where G is not invertible. The resulting fiducial distribution is called a Generalized Fiducial Distribution. To explain the idea we begin with Equation (2) but do not assume that G is invertible with respect to $\boldsymbol{\theta}$. The inverse $G^{-1}(\cdot, \cdot)$ may not exist for one of the following two reasons: for any particular $\mathbf{u}$, either there is no $\boldsymbol{\theta}$ satisfying (2), or there is more than one $\boldsymbol{\theta}$ satisfying (2).

For the first situation, Hannig (2009b) suggests removing the offending values of $\mathbf{u}$ from the sample space and then re-normalizing the probabilities. Such an approach has also been used by Fraser (1968) in his work on structural inference. Specifically, we generate $\mathbf{u}$ conditional on the event that the inverse $G^{-1}(\cdot, \cdot)$ exists. The rationale for this choice is that we know our data $\mathbf{x}$ were generated with some $\boldsymbol{\theta}_0$ and $\mathbf{u}_0$, which implies there is at least one solution $\boldsymbol{\theta}_0$ satisfying (2) when the "true" $\mathbf{u}_0$ is considered. Therefore, we restrict our attention to only those values of $\mathbf{u}$ for which $G^{-1}(\cdot, \cdot)$ exists. However, this set has probability zero in many practical situations leading to non-uniqueness due to the Borel paradox (Casella and Berger, 2002, Section 4.9.3). The Borel paradox is the fact that when conditioning on an event of probability zero, one can obtain any answer.

The second situation can be dealt either by selecting one of the solutions or by the use of the mechanics underlying Dempster-Shafer calculus Dempster (2008). In any case, Hannig (2009a) proves that this non-uniqueness disappears asymptotically under very general assumptions.

Hannig (2009b) proposes the following formal definition of the generalized fiducial recipe. Let $\mathbf{X} \in \mathbb{R}^n$ be a random vector with a distribution indexed by a parameter $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Recall that the data generating mechanism for $\mathbf{X}$ is expressed by (2) where G is a jointly measurable function and $\mathbf{U}$ is a random variable or vector with a completely known distribution independent of any parameters. We define for any measurable set $A \in \mathbb{R}^n$ a set-valued function

$$Q(A, \mathbf{u}) = \{\boldsymbol{\theta} : G(\boldsymbol{\theta}, \mathbf{u}) \in A\}. \tag{3}$$

The function $Q(A, \mathbf{u})$ is the generalized inverse of the function G. Assume $Q(A, \mathbf{u})$ is a measurable function of $\mathbf{u}$.

Suppose that a data set was generated using (2) and it has been observed that the sample value $\mathbf{x} \in A$. Clearly the values of $\boldsymbol{\theta}$ and $\mathbf{u}$ used to generate the observed data will satisfy $G(\boldsymbol{\theta}, \mathbf{u}) \in A$. This leads to the following definition of a generalized fiducial distribution for $\boldsymbol{\theta}$:

$$Q(A, \mathbf{U}^\star) \mid \{Q(A, \mathbf{U}^\star) \neq \emptyset\}, \tag{4}$$

where $\mathbf{U}^\star$ is an independent copy of $\mathbf{U}$.

The object defined in (4) is a random set of parameters (such as an interval or a polygon) with distribution conditioned on the set being nonempty. It is well-defined provided that $P(Q(A, \mathbf{U}^\star) \neq \emptyset) > 0$. Otherwise additional care needs to be taken to interpret this distribution (c.f., Hannig, 2009b). In applications, one can define a distribution on the parameter space by selecting one point out of $Q(A, \mathbf{U}^\star)$.

3 Examples

The following examples provide simple illustrations of the definition of a generalized fiducial distribution.

Example 1. Suppose $\mathbf{U} = (U_1, U_2)$ where U_i are i.i.d. $N(0,1)$ and $\mathbf{X} = (X_1, X_2) = G(\mu, \mathbf{U}) = (\mu + U_1, \mu + U_2)$ for some $\mu \in \mathbb{R}$. So X_i are iid $N(\mu, 1)$. Given a realization $\mathbf{x} = (x_1, x_2)$ of $\mathbf{X}$, the set-valued function Q maps $\mathbf{u} = (u_1, u_2) \in R^2$ to a subset of $\mathbb{R}$ and is given by

$$Q(\mathbf{x}, \mathbf{u}) = \begin{cases} \{x_1 - e_1\} & \text{if} \quad x_1 - x_2 = u_1 - u_2, \\ \emptyset & \text{if} \quad x_1 - x_2 \neq u_1 - u_2. \end{cases}$$

By definition, a generalized fiducial distribution for μ is the distribution of $x_1 - U_1^\star$ conditional on $U_1^\star - U_2^\star = x_1 - x_2$ where $\mathbf{U}^\star = (U_1^\star, U_2^\star)$ is an independent copy of $\mathbf{U}$. Hence a generalized fiducial distribution for μ is $N(\bar{x}, 1/2)$ where $\bar{x} = (x_1 + x_2)/2$.

Example 2. Suppose $\mathbf{U} = (U_1, \ldots, U_n)$ is a vector of i.i.d. uniform $(0,1)$ random variables U_i. Let $p \in [0,1]$. Let $X = (X_1, \ldots, X_n)$ be defined by $X_i = I(U_i < p)$. So X_i are iid Bernoulli random variables with success probability p. Suppose $x = (x_1, \ldots, x_n)$ is a realization of X. Let $s = \sum_{i=1}^n x_i$ be the observed number of 1's. The mapping $Q : [0,1]^n \to [0,1]$ is given by

$$Q(x, \mathbf{u}) = \begin{cases} [0, u_{1:n}] & \text{if} \quad s = 0, \\ (u_{1:n}, 1] & \text{if} \quad s = n, \\ (u_{s:n}, u_{s+1:n}] & \text{if} \quad s = 1, \ldots, n-1 \text{ and } \sum_{i=1}^n I(x_i = 1) I(u_i \leq u_{s:n}) = s, \\ \emptyset & \text{otherwise.} \end{cases}$$

Here $u_{r:n}$ denotes the r^{th} order statistic among $u_1, \ldots, u_n$. So a generalized fiducial distribution for p is given by the distribution of $Q(x, \mathbf{U}^\star)$ conditional on the event $Q(x, \mathbf{U}^\star) \neq \emptyset$. By the exchangeability of $U_1^\star, \ldots, U_n^\star$ it follows that the stated conditional distribution of $Q(x, \mathbf{U}^\star)$ is the same as the distribution of $[0, U_{1:n}^\star]$ when $s = 0$, $(U_{s:n}^\star, U_{s+1:n}^\star]$ for $0 < s < n$, and $(U_{n:n}^\star, 1]$ for $s = n$.

Next, we present a general recipe that is useful in many practical situations.

Example 3. Let us assume that the observations $X_1, \ldots, X_n$ are i.i.d. univariate with distribution function $F(x, \xi)$ and density $f(x, \xi)$, where ξ is a p-dimensional parameter. Denote the generalized inverse of the distribution function by $F^{-1}(\xi, u)$ and use the structural equation

$$X_i = F^{-1}(\xi, U_i) \quad \text{for} \quad i = 1, \ldots, n. \tag{5}$$

If all the partial derivatives of $F(x,\xi)$ with respect to ξ are continuous and the Jacobian

$$\det\left(\frac{\mathbf{d}}{\mathbf{d}\xi}\left(F(x_{i_1},\xi),\dots,F(x_{i_p},\xi)\right)\right)\neq 0$$

for all distinct $x_1,\dots,x_p$, then Hannig (2009a,b) shows that the generalized fiducial distribution (4) is

$$r(\xi)=\frac{f_{\mathbf{X}}(\mathbf{x},\xi)J(\mathbf{x},\xi)}{\int_{\Xi} f_{\mathbf{X}}(\mathbf{x},\xi')J(\mathbf{x},\xi')\,d\xi'}, \tag{6}$$

where $f_{\mathbf{X}}(\mathbf{x},\xi)$ is the likelihood function and

$$J(\mathbf{x},\xi)=\sum_{\mathbf{i}=(i_1,\dots,i_p)}\left|\frac{\det\left(\frac{\mathbf{d}}{\mathbf{d}\xi}\left(F(x_{i_1},\xi),\dots,F(x_{i_p},\xi)\right)\right)}{f(x_{i_1},\xi)\cdots f(x_{i,p},\xi)}\right|. \tag{7}$$

This provides a from of generalized fiducial distribution that is usable in many practical applications, see many of the papers mentioned in introduction. Moreover, if $n=p=1$ (6) and (7) simplify to the Fisher's original definition (1).

Equation (6) is visually similar to Bayes posterior. However, the role of the prior is taken by the function $J(\mathbf{x},\xi)$. Thus unless $J(\mathbf{x},\xi)=k(\mathbf{x})l(\xi)$ where k and l are measurable functions, the generalized fiducial distribution is not a posterior distribution with respect to any prior. A classical example of such a situation is in Grundy (1956).

The quantity $\binom{n}{p}^{-1}J(\mathbf{x},\xi)$ is a U-statistics and therefore it often converges a.s. to

$$\pi_{\xi_0}(\xi)=E_{\xi_0}\left|\frac{\det\left(\frac{\mathbf{d}}{\mathbf{d}\xi}\left(F(X_1,\xi),\dots,F(X_p,\xi)\right)\right)}{f(X_1,\xi)\cdots f(X_p,\xi)}\right|.$$

At first glance $\pi_{\xi_0}(\xi)$ could be viewed as an interesting non-subjective prior. Unfortunately, this prior is not usable in practice, because the expectation in the definition of $\pi_{\xi_0}(\xi)$ is taken with respect to the true parameter ξ_0 which is unknown. However, since $\binom{n}{p}^{-1}J(\mathbf{x},\xi)$ is an estimator of $\pi_{\xi_0}(\xi)$, the generalized fiducial distribution (6) could be interpreted as an empirical Bayes posterior.

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