On purely discontinuous martingales

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Statistics and Probability

2000

Abstract

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Even though the general theory of stochastic processes is a rather well developed field, there is surprisingly little knowledge about the analytical properties of filtrations. In this dissertation we explore connections between purely discontinuous martingales and their filtrations. We are particularly interested in describing the conditions under which there are no non-constant continuous martingales adapted to our filtration.

General martingale theory shows that every martingale can be decomposed into continuous and purely discontinuous parts. In the first part of this dissertation we give a necessary condition on a filtration $\mathcal{F}_t$ implied when the continuous part of the decomposition is 0 $a.s.$ for any $\mathcal{F}_t$ martingale. In the second part of the dissertation we give examples showing that our condition is not sufficient. We also prove various sufficient conditions.

To my wife Marie

ACKNOWLEDGEMENTS

I would like to express my gratitude to my advisor Professor Skorokhod for patience and continuous guidance during the work on this dissertation. I also want to thank the co-chairman of my guidance committee Professor Salehi, who gave me a lot of helpful advice and encouragement during the years I spent at MSU. I really appreciate it. Similarly, I owe my sincere thanks to the other members of my committee Professor Huebner and Professor Gilliland.

I would like to thank the department as a whole for giving me such a great opportunity and much support through the years. Especially, I want to mention Professor Hannan from whom I learned a lot about statistics, mathematics, and who introduced me to the pleasures of physical exercise, Professor Page who was the perfect director of the Statistical Consulting Service, and Professor Levental who taught me probability and mathematical finance. My thanks also goes to my good friend, Dr. White, who was extremely helpful in correcting my English in this work and Professor Štěpán from Charles University, Prague, who first introduced me to the beauty of probability and guided my masters thesis.

I also want to thank my wife who was very courageous to marry a mathematician and is very supportive of my work. I also owe a sincere thanks to my friends at Vineyard Christian Fellowship of Lansing, who became our family away from home. Finally I would like to thank to my LORD, Jesus Christ, who saved me and gave my life direction and purpose.

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List of Notation

We will use standard notation introduced in many probability textbooks (e.g. [13], [34], [16]). The following is a list of some notation used in this dissertation:

$P(X = Y < \infty)$	probability of the set $\{\omega \in \Omega : \ X(\omega) = Y(\omega) < \infty\}$
$\bigvee \mathcal{F}^n$	the smallest filtration (resp. σ-algebra) containing all $\mathcal{F}^n$
$\bigwedge \mathcal{F}^n$	the biggest filtration (resp. σ-algebra) contained in all $\mathcal{F}^n$
X_{t-}	$\lim\limits_{s \uparrow t} X_s$ for a càdlàg process
ΔX	jump process $(X_t - X_{t-})$
$[\tau]$	graph of the stopping time τ
$a \vee b$	maximum of $\{a, b\}$
$a \wedge b$	minimum of $\{a, b\}$
$X \amalg Y$	X and Y are independent
$X \amalg_H Y$	X and Y are conditionally independent given H
$P' \sim P$	$P' \ll P$ and $P' \gg P$
$P' = \rho \cdot P$	measure P' defined by $P'(A) = \int_A \rho dP$
$\mathbb{N}$	set of all natural numbers
$\mathbb{R}_+$	$[0, \infty)$

Introduction

Even though the general theory of stochastic processes is a rather well developed field, there is surprisingly little knowledge about the analytical properties of filtrations. In this dissertation we explore connections between purely discontinuous martingales and their filtrations. We are particularly interested in describing conditions under which there are no non-constant continuous martingales adapted to our filtration.

We assume throughout the whole dissertation that the filtrations satisfy the usual conditions (right continuity and completeness). When we define a filtration we always augment the filtration to the one satisfying the usual conditions without implicitly stating it. Similarly, all martingales are assumed to be in their *càdlàg* version.

In the rest of this chapter we explore the historical background and motivation of our problem. Then we give a list of important definitions, theorems, and notation used in this dissertation.

Chapter 1 contains the main theorem of this dissertation. Under the assumption of quasi-left-continuity we prove that if a filtration $\mathcal{F}_t$ is purely discontinuous (i.e. any $\mathcal{F}_t$-adapted martingale is purely discontinuous), then

$$\mathcal{F}_t = \sigma\{A \cap \{\tau \leq t\}\,; \quad A \in \mathcal{F}_\tau,\ \tau \in T\},$$

where T is a countable collection of totally inaccessible stopping times that exhaust all possible jumps of martingales adapted to $\mathcal{F}_t$. The intuitive meaning of this theorem is that the information contained in our filtration came only from jumps of the martingales. The main difficulty we had to overcome in the proof of this theorem was related to the fact that we allowed infinitely many jumps on finite intervals.

Chapter 2 contains several examples showing that the behavior of a purely discontinuous filtration can be sometimes non-intuitive. In particular we show that the necessary condition proved in Chapter 1 is not sufficient. We also show that a subfiltration of a purely discontinuous filtration does not have to be purely discontinuous.

In Chapter 3 we prove that a filtration is purely discontinuous if and only if our probability measure is an extreme point in the set of all probabilities that preserve all compensators. The proof of this theorem is almost identical to the proof of the characterization theorem for the weak representation property. The main drawback of this condition is that it is usually very hard to verify.

Finally, Chapter 4 contains several sufficient conditions for a filtration to be purely discontinuous. These conditions, derived as analytical properties of the filtration, are easier to verify than the condition of Chapter 3. We also give several examples of purely discontinuous filtrations, the most important of them being a filtration generated by a purely discontinuous Levy process. The last theorem of this chapter deals with discrete time sequences and gives a result on an equivalent change of the probability measure under which the sequence becomes quasi-Markov. We hope to apply this theorem to get another sufficient condition for purely discontinuous filtrations in the future.

The main problem that still remains open is to find a necessary and sufficient condition using the analytical properties of the filtration.

0.1 Historical Remarks

Martingales are one of the most important objects in the modern theory of probability. The term martingale (originally denoting part of horse's harness and later used for a special gambling system) was introduced into the probability theory in the first half of 20th century as a natural generalization of sums of independent random variables, by Bernstein (1927, 1937), Lévy (1937), and Ville (1939). The basic regularity theorems for continuous time martingales first appear in a paper by Doob (1951).

Increasing sequences of σ-algebra have been already used by Doob around 1940. They were also used in the work of Itô (1944, 1946) and others. In their full generality, filtrations first appear in the famous book, *Stochastic Processes*, by Doob (1953). The idea of systematically extending to the stopping times results that are valid for fixed times was inspired by the strong Markov property, first mentioned by Doob (1942) in a paper on Markov chains. We also owe a lot of useful results on stopping times to the school of Dynkin. Systematic study of stopping times and their associated σ-algebras was initiated by a paper of Chung and Doob (1965).

Predictability is a very clear concept in discrete time but it is somewhat unintuitive in continuous time. Predictable stopping times appear implicitly in works of Blumethal and Hunt (1957–1958). The theory of predictable and optional σ-algebras was developed into the modern theory of stochastic processes by Meyer (1966), Dellacherie (1972) and others. Let us note that this theory was motivated by Markov processes. The importance of predictable σ-algebra became clear after Doléans (1967) had proved the equivalence between natural and predictable increasing processes, thereby establishing the ultimate version of Doob-Meyer decomposition.

A compensator of a stopping time can be defined as the predictable part of Doob-Meyer's decomposition of the process $1_{\{\tau \leq t\}}$. Compensators for more general objects, such as random measures, were obtained by Jacod (1975). The compensators for random measures can be derived using the theory of dual predictable projections.

In 1976, Yœurp proved the existence of an orthogonal decomposition of martingales into continuous and purely discontinuous parts. Meyer (1976) then proved that the purely discontinuous part is a sum of compensated jumps. The term, purely discontinuous martingale, is misleading. It does not refer to a piecewise constant process. In fact there is a purely discontinuous martingale that has non-constant continuous trajectories with non-zero probability. Rather the term, purely discontinuous martingale, denotes the "non-continuous" part of the orthogonal decomposition. Incidentally, any purely discontinuous martingale is orthogonal (in the sense of quadratic variation) to all continuous martingales. In particular, any martingale with locally bounded variation is purely discontinuous.

The main object of my dissertation is to characterize filtrations for which the continuous part of the decomposition is constant almost surely. The problem was motivated by an effort to generalize a notion of jumping filtration to admit any purely discontinuous martingale.

A filtration $\mathcal{F}_t$ is called jumping if it is generated by an increasing step process.

Equivalently,

$$\mathcal{F}_t = \sigma\{A \cap \{\tau_k \le t\}\,; \quad A \in \mathcal{F}_{\tau_k},\ k \in \mathbb{N}\},$$

where $\tau_k \to \infty$ is an increasing sequence[1] of stopping times (the times of jumps of the step process). The main feature of this filtration is that it is constant on $[\tau_k, \tau_{k+1})$. A well known example of a jumping filtration is the natural filtration of a Poisson process.

Many interesting analytical properties of jumping filtrations were studied in a series of papers by He and Wang in the early 1980s. An excellent summary of the results on jumping filtrations can be found in their book, *Semimartingales and Stochastic Calculus* (1992). An important characterization that directly motivated my research is due to Jacod and Skorokhod (1994). They prove, in full generality, that a filtration is jumping if and only if any adapted martingale is a process of locally bounded variation. Thus, jumping filtrations support no non-constant continuous martingales.

The limiting feature of jumping filtrations is that they allow only for a finitely many jumps on finite intervals. It is well-known that there exist purely discontinuous martingales that have infinitely many jumps on finite intervals. We will call this phenomena accumulation of jumps. An example of such a martingale is a Gamma process (a particular purely discontinuous Levy process). The main theorem of this dissertation gives a necessary condition for a filtration to support only purely discontinuous martingales. This condition is very similar to the definition of jumping filtration. We are able to accommodate all types of purely discontinuous processes

[1]It is possible that $P(\tau_k = \infty) > 0$.

by allowing for the accumulation of jumps.

Sufficient conditions for a filtration to be purely discontinuous (i.e. support no non-constant continuous martingale) are closely related to the weak predictable representation property for martingales. A process X has weak predictable representation property if any local martingale can be written in the form

$$M = M_0 + H \cdot X^c + W * \tilde{\mu},$$

where X^c is the continuous part of X, $\tilde{\mu}$ is the compensated random measure associated with jumps of X, and the integrators are predictable. Hence if $X^c = 0$ then any martingale is a stochastic integral with respect to the compensated jump measure $\tilde{\mu}$ and therefore it is purely discontinuous.

Random measures and the associated integrals were introduced by Itô (1951). They were studied by Skorokhod (1965) in the case when the compensator is deterministic. The general theory was developed by Jacod (1976). The notion of predictable representation property is due to Jacod (1977), where he also proved that the process X has a weak predictable representation property if and only if the probability measure P is an extreme point in a certain set of probability measures. We apply the techniques used to prove this theorem in Chapter 3.

0.2 Theory of Stochastic Processes – Overview

The purpose of this section is to give a reader an overview of the main definitions and results that were used in this dissertation.

0.2.1 Filtrations, stopping times, martingales

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Unless stated otherwise, all objects will be defined on this probability space. A *filtration* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is a family of increasing sub-σ-algebras of the σ-algebra $\mathcal{F}$. Denote $\mathcal{F}_\infty = \bigvee_{t>0} \mathcal{F}_t$.

A filtration is *right-continuous* if, for any t, $\mathcal{F}_t = \bigcap_{t<s} \mathcal{F}_s$. A filtration is *$P$-complete* if $\mathcal{F}_0$ contains all P-null sets. We assume throughout this dissertation that the filtrations at hand are right-continuous and complete (called the *usual conditions*).

A random variable T is a *stopping time* if

$$\{T \le t\} \in \mathcal{F}_t, \quad \text{for all } t \ge 0.$$

A stopping time T is *predictable* if there is an non-decreasing sequence of stopping times $T_n \to T$, such that $T_n < T$ on the set $\{T > 0\}$. The sequence T_n is called an *announcing sequence*. Stopping time T is *totally inaccessible* if $P(T = H < \infty) = 0$ for any predictable stopping time H.

Let T be a stopping time. Define σ-algebras

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty : \ A \cap \{\tau \le t\} \in \mathcal{F}_t\}, \quad \mathcal{F}_{T-} = \sigma\{A \cap \{\tau > t\} : \ A \in \mathcal{F}_t\}.$$

Clearly $\mathcal{F}_{T-} \subset \mathcal{F}_T$, and σ-algebra $\mathcal{F}_T$ is, intuitively, the knowledge at time T. Similarly, $\mathcal{F}_{T-}$ is the knowledge immediately preceding time T. It can be shown that if $T_n \downarrow T$ then $\mathcal{F}_T = \bigwedge \mathcal{F}_{T_n}$. Similarly, if T is a predictable stopping time and T_n is its announcing sequence then $\mathcal{F}_{T-} = \bigvee \mathcal{F}_{T_n}$

A filtration $\mathcal{F}$ is *quasi-left-continuous* if $\mathcal{F}_{T-} = \mathcal{F}_T$ for any predictable stopping time T.

A process M_t is *adapted* if M_t is $\mathcal{F}_t$ measurable. An adapted process is called a *martingale* if $E[M_t \,|\, \mathcal{F}_s] = M_s$ for any $s < t$. Similarly, an adapted process is called a *submartingale* if $E[M_t \,|\, \mathcal{F}_s] \geq M_s$. Notice that the notion of martingale depends on the underlying filtration. It is possible to have a process that is adapted to two different filtrations, but is a martingale with respect to one and is not a martingale with respect to the other. However, the following is clearly true.

Proposition 0.1. *Let X be a martingale for the filtration cF_t. Let $\mathcal{G}_t$ be a subfiltration of cF_t (i.e. $\mathcal{G}_t \subset \mathcal{F}_t$), such that X is adapted to $\mathcal{G}_t$. Then X is a martingale for $\mathcal{G}_t$.*

A process M_t is a *local martingale* if there is an increasing sequence of stopping times $T_n \uparrow \infty$ such that $M_{t \wedge T_n}$ is a martingale for each n. Such a sequence of stopping times is called *localizing sequence.* A local martingale does not have to be a martingale. *Locally bounded variation, locally integrable, etc.* are defined similarly.

Proposition 0.2. *Let M_t be a submartingale. Then M_t has a càdlàg (right continuous, left limits) version if and only if the function $t \to EX_t$ is right continuous.*

For a proof see [34] Theorem 6.27. (Remember that the filtration is assumed to satisfy the usual conditions.)

We assume that all martingales are in their càdlàg version.

Proposition 0.3. *If M_t is a continuous local martingale of locally finite variation, then $M = M_0$ a.s.*

For a proof see [34] Proposition 15.2.

Proposition 0.4. *Filtration $\mathcal{F}$ is quasi-left-continuous if and only if $\Delta M_\tau = 0$ a.s. for any martingale M and any predictable stopping time τ.*

For a proof see [34] Proposition 22.19.

The last notion we introduce in this section is *predictable σ-algebra*. Let $\mathcal{P}$ be a σ-algebra in $\mathbb{R}_+ \times \Omega$ generated by all continuous adapted processes. The sets in $\mathcal{P}$ are called *predictable sets*, and the $\mathcal{P}$ measurable functions on $\mathbb{R}_+ \times \Omega$ are called *predictable processes*.

Lemma 0.5. *For any stopping time τ and predictable process X_t, the random variable $X_\tau 1_{\{\tau<\infty\}}$ is $\mathcal{F}_{\tau-}$-measurable. Conversely, if ξ is a real $\mathcal{F}_{\tau-}$-measurable random variable, then there exists a predictable process X_t such that $\xi 1_{\{\tau<\infty\}} = X_\tau 1_{\{\tau<\infty\}}$.*

For a proof see [16] Corollary 3.23.

The predictable processes play a very important role as compensators and integrands of stochastic integrals.

Proposition 0.6. *A local martingale is predictable if and only if it is continuous.*

For a proof see [34] Proposition 22.16.

0.2.2 Decomposition theorems

Theorem 0.7. *(Doob-Meyer) A process X is a local submartingale if and only if $X = M + A$, where M is a local martingale and A is a locally integrable, increasing, predictable process. In that case M and A are a.s. unique.*

For a proof see [34] Theorem 22.5.

This theorem is very profound and we will use it to define both compensators and predictable square characteristics of martingales.

Suppose τ is a stopping time. The process $X_t = 1_{\{\tau \le t\}}$ is a submartingale. Thus there is a predictable increasing process C_t such that $X_t - C_t$ is a local martingale. (It can be shown $X_t - C_t$ is a martingale and C_t is an integrable process.) The process C_t is called a *compensator* of τ.

Proposition 0.8. *Let τ be a stopping time with compensator C_t. Then τ is totally inaccessible if and only if C_t is continuous.*

For a proof see [34] Corollary 22.18.

We say that a martingale M_t is *square integrable* if $\sup_t EM_t^2 < \infty$. Denote by $\mathcal{M}^2$ the collection of all square integrable martingales. It is not difficult to see that square integrable martingales are also uniformly integrable. Also, the space $\mathcal{M}^2$, endowed with inner product $(M, N) = EM_\infty N_\infty$, is a Hilbert space. We say that M and N are *weakly orthogonal* if $(M, N) = 0$.

If M is a locally square integrable martingale, then the process M_t^2 is a local submartingale. Let $\langle M \rangle_t$ denote the *predictable quadratic variation,* the predictable part of the Doob Meyer decomposition of M_t^2. If M, N are both locally square integrable, then the *predictable covariation* is

$$\langle M, N \rangle = \frac{1}{4} \left(\langle M + N \rangle - \langle M - N \rangle \right).$$

The process $\langle M, N \rangle$ is a predictable process of locally integrable variation. Moreover, $MN - \langle M, N \rangle$ is a locally square integrable martingale. We say that M, N are *orthogonal* if $\langle M, N \rangle = 0$.

We say that a martingale is *purely discontinuous* if it is a sum of compensated jumps. (It is not necessarily a process of bounded variation.)

Theorem 0.9. *(Yœurp, Meyer) Suppose M is a local martingale. Then M can be uniquely decomposed as*

$$M = M_0 + M^c + M^d,$$

where M^c is a continuous local martingale starting at 0, and M^d is a purely discontinuous martingale. Furthermore, if M is locally square integrable then $\left\langle M^c, M^d \right\rangle = 0$.

For a proof see [16] Chapter 7. (We have pooled several theorems from that chapter to get Theorem 0.9.)

Proposition 0.10. *Let M be a local martingale. Then for all $t \geq 0$*

$$\sum_{s \leq t} (\Delta M_s)^2 < \infty \quad a.s.$$

For a proof see [16] Lemma 7.27.

Let M, N be local martingales. Proposition 0.10 and the fact that any continuous process starting at 0 is locally bounded allow us to define *quadratic covariation* by

$$[M,N] = M_0 N_0 + \langle M^c, N^c \rangle + \sum \Delta M_s \Delta N_s.$$

It is easy to see that $[M,N]$ is an adapted process of finite variation, and if $\langle M,N \rangle$ exists then $[M,N] - \langle M,N \rangle$ is a local martingale.

Theorem 0.11. *Suppose $\mathcal{F}_\infty$ is countably generated. Then, there exists a sequence of orthogonal square integrable martingales ξ_n such that for any square integrable martingale M starting at 0 there is a sequence of predictable processes H_n such that*

$$M = \sum_n \int H_n d\xi_n. \tag{1}$$

(The limit in (1) is considered in $\mathcal{M}^2$. The integrals are considered as stochastic integrals[2].) For the proof, see [7] VIII.46 — VIII.52, and use the fact that the space $L^2(\mathcal{F}_\infty)$ is separable and therefore has a countable basis.

0.2.3 Jumping filtration

We say that a filtration $\mathcal{F}$ is *jumping* if there is a sequence of stopping times τ_n increasing a.s. to ∞ such that

$$\mathcal{F}_t = \mathcal{F}_{\tau_n} \; a.s. \quad \text{on the set } \{\tau_n \le t < \tau_{n+1}\}.$$

[2]The theory of stochastic integration is an essential part of the theory of stochastic processes. However, since it is not an essential part of this dissertation we omit it. An interested reader can find it for example in [7], [16], or any other book on theory of stochastic processes.

Equivalently,

$$\mathcal{F}_t = \sigma\{A \cap \{\tau_k \le t\}\,; \quad A \in \mathcal{F}_{\tau_k},\ k \in \mathbb{N}\}.$$

Theorem 0.12. *(Jacod, Skorokhod) A filtration is jumping if and only if all its martingales are a.s. of locally bounded variation.*

For the proof see [29].

Proposition 0.13. *Let τ be a non-negative random variable and define $\mathcal{F}_t = \sigma\{\{\tau \le s\} : \ s \le t\}$[3]. Then τ is totally inaccessible if and only if the distribution of τ is continuous on $[0,\infty)$ (with a possible atom $\{\tau = \infty\}$). In this case the filtration $\mathcal{F}$ is quasi-left-continuous.*

For a proof see [16] Example 5.70.

0.2.4 Miscellaneous

Define a *graph* of a stopping time τ as

$$[\tau] = \{(t,\omega) \in [0,\infty) \times \Omega : \ t = \tau(\omega)\}.$$

We say that H is a *thin* process if there is a sequence of stopping times $\{\tau_n\}$ such that

$$[H \neq 0] = \bigcup_{n \in \mathbb{N}} [\tau_n].$$

Let $(\Delta M)_t = M_t - M_{t-}$ denote the *jump process* of any càdlàg process M.

[3]Remember that we assume that $\mathcal{F}$ is automatically augmented to satisfy the usual conditions.

Theorem 0.14. *In order that a thin process H be the jump process ΔM of a local martingale M it is necessary and sufficient that*

1. $E[H_\tau \,|\, \mathcal{F}_{\tau-}] = 0$ *a.s. for any predictable stopping time* τ.

2. $\sqrt{\sum_{s\leq t} H_s^2}$ *is an increasing locally integrable process.*

Additionally, M exists as a martingale of locally integrable variation if and only if 2. is replaced by the condition: $\sum_{s\leq t}|H|$ is an increasing locally integrable process.

For a proof see [16] Theorem 7.42 and Corollary 7.43.

A *semimartingale* is a right-continuous adapted process admitting a decomposition $M+A$, where M is a local martingale and A is a process of locally integrable variation.

Theorem 0.15. *(Doléans' exponential) For any semimartingale X with $X_0 = 0$, the equation $Z = 1 + Z_- \cdot X$ has the a.s. unique solution*

$$Z_t = \mathcal{E}(Y) \equiv \exp(X_t - \frac{1}{2}\langle X^c \rangle_t) \prod_{s\leq t}(1+\Delta X_s)e^{-\Delta X_s}.$$

For a proof see [34] Theorem 23.8.

Theorem 0.16. *(General Girsanov's Theorem) Assume that $P' \ll P$, define $Z_t = E[\frac{dP'}{dP} \,|\, \mathcal{F}_t]$, and consider a local P-martingale M such that the process $[M,Z]$ has locally integrable variation and P-compensator $\langle M, Z\rangle$. Then $\tilde{M} = M - \frac{1}{Z_-} \cdot \langle M, Z\rangle$ is a local P'-martingale. Moreover, if M is continuous then M' is continuous, and $\langle M\rangle$ calculated under P equals $\langle M'\rangle$ calculated under P' a.s. with respect to P'.*

For a proof see [34] Theorem 23.9, [16] Corollary 12.15, and Corollary 12.16.

Let ζ_n be a discrete time stationary sequence and let $\mathcal{G}$ be its shift-invariant σ-algebra. Denote Θ the shift operator.

Theorem 0.17. *(Birkhoff) Consider a measurable function $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ such that $E|f(\zeta)|^p < \infty$ for some $p \geq 1$. Then*

$$n^{-1} \sum_{k<n} f(\Theta^k \zeta) \to E[f(\zeta) \,|\, \mathcal{G}] \quad \textit{a.s. and in } L^p.$$

For a proof see [34] Theorem 9.6.

We say that X is *conditionally independent* of Y given H $(X \amalg_H Y)$ if

$$P[(X \in A) \cap (Y \in B) \,|\, H] = P[(X \in A) \,|\, H].P[(Y \in B) \,|\, H].$$

Proposition 0.18. *We have $X \amalg_H Y$ if and only if*

$$P[X \in A \,|\, H, Y] = P[X \in A \,|\, H].$$

For a proof see [34] Proposition 5.6.

Chapter 1

Main Theorem

In this chapter we state and prove the main theorem of this dissertation.

Unless stated otherwise we always assume that the filtration $\mathcal{F}_t$ is complete, right-continuous, quasi-left-continuous, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ *a.s.*, and the σ-algebra $\mathcal{F}_\infty$ is countably generated. All martingales are considered to be in *càdlàg* version.

Let us introduce the following definitions.

Definition 1.1. A filtration is called purely discontinuous if any continuous adapted martingale is constant *a.s.*

Definition 1.2. Let T be a countable collection of stopping times. Then we define

$$S_\tau = \inf\{\upsilon(\omega);\ \upsilon(\omega) > \tau(\omega), \upsilon \in T\},$$

where τ is a stopping time, and

$$A_t(\omega) = \sup\{\tau(\omega);\ \tau(\omega) < t, \tau \in T\},$$

where t is a deterministic time.

Note that S_τ is a stopping time, while A_t is not. The random variables S_t and A_t will be often referred to as "first jump after τ" and "last jump before t" respectively. This comes from the observation that if the set T is the set of all possible jumps of adapted martingales then any $\mathcal{F}_t$ martingale does not have jumps on the interval (τ, S_τ) and has at least one jump on $[S_\tau, S_\tau + \varepsilon)$ for all $\varepsilon > 0$. An analogous statement is true for A_t.

Now we can state the main theorem of this dissertation.

Theorem 1.1. *Let $\mathcal{F}_t$ be a purely discontinuous filtration. Then*

$$\mathcal{F}_t = \sigma\{A \cap \{\tau \leq t\}\,; \quad A \in \mathcal{F}_\tau,\ \tau \in T\}, \tag{1.1}$$

where T is a countable collection of totally inaccessible stopping times with disjoint graphs.

The intuitive meaning of this theorem is that the information contained in our filtration came only from jumps of the martingales. To prove it we will need the following lemma.

Lemma 1.2. *The following is true under the assumptions of Theorem 1.1: Let $H \leq S$ be two stopping times. If any $\mathcal{F}_t$ martingale is continuous on the interval (H, S) then $\mathcal{F}_t = \mathcal{F}_H$ on $\{H \leq t < S\}$ (i.e. for every $A \in \mathcal{F}_t$ there is $A' \in \mathcal{F}_H$ such that $A \cap \{H \leq t < S\} = A' \cap \{H \leq t < S\}$ a.s.).*

Proof. The proof of the lemma is due to Jacod and Skorokhod [29]. We corrected a little typo and checked that it works in our situation.

Fix t and choose $A \in \mathcal{F}_t$. Define the martingale $N^A_s = P[A \cap \{H \le t < S\} \,|\, \mathcal{F}_s]$, and the point process $X_s = 1_{\{S \le s\} \cap \{H < S\}}$ with compensator Y. Notice that

$$B = A \cap \{H \le t < S\} \in \mathcal{F}_{S-} \wedge \mathcal{F}_t. \tag{1.2}$$

Hence the martingale

$$M_t = N^A_s - N^A_{s\wedge H}$$

is null on $[0, H]$ and a constant on $[S \wedge t, \infty)$. Assumptions of the lemma imply $[\Delta M] \subset [S]$. Calculate

$$\Delta M_S = \Delta N^A_S = -N^A_{S-} 1_{\{t \ge S\}}. \tag{1.3}$$

The last equality in (1.3) follows from the following facts: $\Delta N^A_S = 0$ on the set $\{S > t\}$ and the $N^A_S = 0$ on the set $\{S \le t\}$. Hence

$$M_s = \int_0^s H_u d(X - Y), \tag{1.4}$$

where $H_u = -N^A_{u-} 1_{\{t \ge u\}}$ is a bounded predictable process. Formula (1.4) then implies:

$$N^A_s = N^A_H + \int_0^{s \wedge t} N_{u-} dY_u \ a.s. \quad \text{if } H \le t < S.$$

Now observe that $Y_s = 0$ for $s \leq H$ and therefore $N_s^A = N_H^A \mathcal{E}(Y)_{s\wedge t}$ if $H \leq t < S$, where $\mathcal{E}(y)$ denotes the Doléans' exponential of Y (see Theorem 0.15). Since A was an arbitrary set we get $N_s^\Omega = N_H^\Omega \mathcal{E}(Y)_{s\wedge t}$ if $H \leq t < S$. This implies

$$N_t^A N_H^\Omega = N_t^\Omega N_H^A \; a.s. \quad \text{on } \{H \leq t < S\}. \tag{1.5}$$

Define an $\mathcal{F}_H$ measurable set $A' = \{N_H^\Omega = N_H^A > 0\}$. The relation (1.5), $N_t^A = 1_{A\cap\{H\leq t<S\}}$, and $N_t^\Omega = 1_{\{H\leq t<S\}}$ imply that $A \cap \{H \leq t < S\} = A' \cap \{H \leq t < S\}$ $a.s.$ $\square$

Proof of Theorem 1.1. First we construct the set T. Let M be a fixed square integrable martingale with $M(0) = 0$. Define stopping times

$$\tau_0^n = 0,\; \tau_k^n = \inf\{t > \tau_{k-1}^n; |\Delta M| \in \left(\frac{1}{n}, \frac{1}{n-1}\right]\}.$$

Set $T_M = \{\tau_k^n;\; \tau_k^n \neq \infty \text{ a.s.}\}$. The definition of τ_k^n assures that if $\tau, \upsilon \in T_M$ are two different stopping times, then the set $\{\tau = \upsilon < \infty\} = \emptyset$.

Since the filtration is quasi-left-continuous, the stopping times τ_k^n are totally inaccessible. It is known that (see Theorem 0.9)

$$M = \sum_{\tau\in T_M} \Delta M_\tau 1_{\{\tau\leq t\}} - C_{t,\tau}, \tag{1.6}$$

where the right-hand-side of (1.6) is a compensated sum of jumps and the compensators $C_{t,\tau}$ are continuous functions of finite variation.

Since $\mathcal{F}_\infty$ is countably generated, we can find a countable set of square integrable martingales $\{\eta_n\}$ such that $\langle \eta_n, \eta_m \rangle = 0$ and each $M \in \mathcal{M}^2$ starting at 0 can be written as $M = \sum_n \int V_n d\eta_n$ (see Theorem 0.11). We define $\widetilde{T} = \bigcup_n T_{\eta_n}$.

Because $\widetilde{T}$ is countable, we can order the elements of $\widetilde{T}$ to form a sequence $\{\tau_n\}$. The jumping process $N_t = \sum_n 2^{-n} 1_{\{\tau_n \le t\}}$ is bounded. Therefore the martingale $M_t = N_t - C_t$ is square integrable and the compensator C_t is continuous. Thus $[\Delta M] = [\Delta N] = \bigcup_{\tau \in \widetilde{T}} [\tau]$. Define $T = T_M$. Clearly T contains stopping times with mutually disjoint graphs and $\bigcup_{\tau' \in \widetilde{T}} [\tau'] = \bigcup_{\tau \in T} [\tau]$.

Let $\mathcal{G}_t = \sigma\{A \cap \{\tau \le t\}; A \in \mathcal{F}_\tau, \tau \in T\}$. It is easy to see that $\mathcal{G} \subset \mathcal{F}$. To prove Theorem 1.1 we need to prove $\{\mathcal{F}_t\} = \{\mathcal{G}_t\}$.

To begin with we prove that for any totally inaccessible $\mathcal{F}_t$ stopping time υ the filtrations $\mathcal{G}_\upsilon = \mathcal{F}_\upsilon$ on the set $\{\upsilon < \infty\}$.

If $\upsilon \in T$, the assertion follows from the definition. Let us assume that $\upsilon \notin T$. The $\mathcal{F}_t$ martingale $X_t = 1_{\{\upsilon \le t\}} - C_t$ is square integrable, hence $X = \sum \int V_n d\eta_n$ *a.s.*. It follows that $\Delta X = \sum V_n \cdot \Delta \eta_n$ *a.s.*. From this we get $[\upsilon] = \bigcup_{\{\tau \in T\}} \{\tau = \upsilon\} \cap [\tau]$. Thus for any finite $t \in \mathbb{R}$, and for any $A \in \mathcal{F}_\upsilon$

$$A \cap \{\upsilon \le t\} = \bigcup_{\{\tau \in T\}} A \cap \{\tau = \upsilon\} \cap \{\tau \le t\} \in \mathcal{G}_t.$$

It follows that υ is a stopping time with respect to $\mathcal{G}_t$ and $\mathcal{G}_\upsilon = \mathcal{F}_\upsilon$ on the set $\{\upsilon < \infty\}$. This restriction arises from the definition of $\mathcal{G}_\infty = \bigvee \mathcal{G}_t = \sigma\{A \cap \{\tau < \infty\}; A \in \mathcal{F}_\tau, \tau \in T\}$.

The following simple observations are valid for any sequence of stopping times

$\{\tau_n\}$. If $\mathcal{F}_{\tau_n} = \mathcal{G}_{\tau_n}$ on $\{\tau_n < \infty\}$ and $\tau_n \downarrow \tau$, then

$$\mathcal{F}_\tau = \bigwedge \mathcal{F}_{\tau_n} = \bigwedge \mathcal{G}_{\tau_n} = \mathcal{G}_\tau \text{ on } \bigcup \{\tau_n < \infty\} = \{\tau < \infty\}.$$

Similarly if $\tau_n \uparrow \tau$, then

$$\mathcal{F}_\tau = \bigvee \mathcal{F}_{\tau_n} = \bigvee \mathcal{G}_{\tau_n} = \mathcal{G}_\tau \text{ on } \bigcap \{\tau_n < \infty\} = \{\tau < \infty\}.$$

The latter statement is true because the filtrations are quasi-left-continuous.

Recall that in the Definition 1.2 we have defined:

$$S_\tau = \inf\{\upsilon_{\{\upsilon > \tau\}}; \upsilon \in T\},$$

and

$$A_t(\omega) = \sup\{\tau(\omega); \tau(\omega) < t, \tau \in T\}.$$

The random variable S_τ is a stopping time, while A_t is not. However A_t is a $\mathcal{G}_t$ measurable random variable. This follows from the relation:

$$\{A_t \leq s\} = \{S_s \geq t\} \in \mathcal{G}_t.$$

Since T is a countable set, it follows from the previous statements that $\mathcal{F}_{S_\tau} = \mathcal{G}_{S_\tau}$ on the set $\{S_\tau < \infty\}$. Similarly, if ζ is a predictable stopping time and the set $\{\zeta = A_t \leq t\}$ has a non-zero probability, $\mathcal{F}_\zeta = \mathcal{G}_\zeta$ on the set containing $\{\zeta = A_t\}$.

To prove the latter note that $\mathcal{F}_0 = \mathcal{G}_0$ is a trivial σ-algebra. We assume without loss of generality that $\zeta > 0$ $a.s.$, and $\zeta_n < \zeta$ is a sequence announcing ζ. It follows from the definition of A_t that $S_{\zeta_n}(\omega) < A_t(\omega) \leq t$ on the set $\{\zeta = A_t\}$. The sequence S_{ζ_n} is nondecreasing, so we can define $S = \lim S_{\zeta_n}$. Noticing that S is a $\mathcal{G}_t$ stopping time we deduce that $\mathcal{F}_S = \mathcal{G}_S$ on the set $\{S < \infty\}$. The statement is implied by the fact that $\{\zeta = A_t\} \subset \{\zeta = S < \infty\}$.

To finish the proof it will be enough to prove $\mathcal{F}_{t_0} = \mathcal{G}_{t_0}$ for a fixed t_0. I will do it separately on three different $\mathcal{G}_{t_0}$ measurable sets.

Following for a moment the proof of Proposition 22.4 in [34] let

$$p = \sup P\left(\bigcup_n \{A_{t_0} = \tau_n^{\mathsf{p}} < \infty\}\right),$$

where the supremum extends over all possible sequences of predictable stopping times. Combining sequences such that the probability on the right-hand-side approaches p we construct a sequence of predictable stopping times for which the supremum is attained. Let $\{\tau_n^{\mathsf{p}}\}$ be this sequence. (Note that if $p = 0$ this sequence is empty.) Define the following sets:

$$B_1 = \{A_{t_0} = t_0\} \cup \{S_{t_0} = t_0\}$$

$$B_2 = \left(\bigcup_n \{A_{t_0} = \tau_n^{\mathsf{p}}\} \cup \bigcup_{\tau^{\mathsf{na}} \in T} \{A_{t_0} = \tau^{\mathsf{na}}\}\right) \setminus B_1$$

$$B_3 = \Omega \setminus (B_1 \cup B_2)$$

These sets are $\mathcal{G}_{t_0}$ measurable. This is clearly true for B_1. The set B_2 is $\mathcal{G}_{t_0}$ measurable because all the stopping times involved in its definition can be taken to be $\mathcal{G}_t$ stopping times.

It follows from the previous discussion that $\mathcal{F}_{t_0} = \mathcal{G}_{t_0}$ on the set B_1.

Denote the sequence of stopping times that was involved in definition of B_2 by $\{\nu_n\}$. We have established, that we can choose this sequence so that $P(\nu_n = A_{t_0} < \infty) > 0$, and $\mathcal{F}_{\nu_n} = \mathcal{G}_{\nu_n}$ on the set $\{\nu_n < \infty\} \supset \{\nu_n = A_{t_0}\}$. (The statement is true for both totally inaccessible and predictable stopping times.) As mentioned before no $\mathcal{F}_t$ martingale has jumps between times ν_n and S_{ν_n} for any fixed n. It follows from Lemma 1.2 that if $B \in \mathcal{F}_{t_0}$ there is $B' \in \mathcal{F}_{\nu_n} = \mathcal{G}_{\nu_n}$ such that

$$B \cap \{\nu_n \leq t < S_{\nu_n}\} = B' \cap \{\nu_n \leq t < S_{\nu_n}\} \in \mathcal{G}_t.$$

Finally $B_2 \subset \bigcup_n \{\nu_n \leq t < S_{\nu_n}\}$ implies $\mathcal{F}_{t_0} = \mathcal{G}_{t_0}$ on B_2.

To overcome problems associated with B_3 we will enlarge our filtrations. Define

$$\tilde{A} = \begin{cases} A_{t_0} & \text{on the set } B_3, \\ \infty & \text{otherwise.} \end{cases}$$

Note that $\tilde{A}$ is $\mathcal{G}_{t_0}$ measurable random variable, and $P(\tilde{A} = \tau) = 0$ for all $\mathcal{F}_t$ stopping times τ. We further define[1]

$$\mathcal{H}_s = \sigma\{\tilde{A} \leq x;\, x \leq s\}, \quad \tilde{\mathcal{F}}_s = \mathcal{F}_s \vee \mathcal{H}_s, \quad \tilde{\mathcal{G}}_s = \mathcal{G}_s \vee \mathcal{H}_s.$$

[1] The filtrations are enlarged to satisfy the usual conditions where necessary.

The filtration was augmented just enough to make the random variable $\tilde{A}$ a stopping time. First we prove that it is a totally inaccessible stopping time.

Let τ be a $\tilde{\mathcal{F}}_t$ stopping time. I will prove that there is a $\mathcal{F}_t$ stopping time τ', such that $\tau = \tau'$ on the set $\{\tau < \tilde{A}\}$. Denote $C_s = \{\tau > s\} \cap \{\tilde{A} > s\} = \{\tau \wedge \tilde{A} > s\}$. Since $\{\tilde{A} > s\}$ is an atom in $\mathcal{H}_s$, there is $\bar{D}_s \in \mathcal{F}_s$ such that $C_s = \bar{D}_s \cap \{\tilde{A} > s\}$. We define

$$D_s = \bigcup_{\substack{q_1 > s \\ q_1 \in \mathbb{Q}}} \bigcap_{\substack{q_2 \leq q_1 \\ q_2 \in \mathbb{Q}}} \bar{D}_{q_2}.$$

The right-continuity of the filtrations involved gives $D_s \in \mathcal{F}_s$, and the definition of C_s gives $C_s = D_s \cap \{\tilde{A} > s\}$. Define $\tau'(\omega) = \sup\{t : \, \omega \in D_t\}$. It is a $\mathcal{F}_t$ stopping time and $\tau = \tau'$ on the set $\{\tau < \tilde{A}\}$. The fact that $\tilde{A}$ is a totally inaccessible $\tilde{\mathcal{F}}_t$ stopping time follows directly.

In a similar way we prove that for any $\tau \in T$, $\mathcal{F}_{\tau-} = \tilde{\mathcal{F}}_{\tau-}$. Namely, let $t < t_0$ and $B \in \tilde{\mathcal{F}}_t$,

$$B \cap \{\tau > t\} = (B \cap \{t < \tau \leq t_0\}) \cup (B \cap \{t_0 < \tau\}) \in \mathcal{F}_{\tau-},$$

since $B \in \mathcal{F}_t \subset \mathcal{F}_{t_0}$ and $\{t < \tau \leq t_0\} \subset \{\tilde{A} > t\}$.

As a next step we want to prove that $\tilde{\mathcal{F}}_{\tilde{A}} = \tilde{\mathcal{G}}_{\tilde{A}}$ on the set $\{\tilde{A} < \infty\}$. For any $B \in \mathcal{F}_\tau$, $\tau \in T$

$$B \cap \{\tau < \tilde{A}\} = \bigcup_{q \in \mathbb{Q}} B \cap \{\tau \leq q < \tilde{A}\} \in \tilde{\mathcal{G}}_{\tilde{A}-}.$$

Calculate on the set $\{\tilde{A} < \infty\}$

$$
\begin{aligned}
\tilde{\mathcal{F}}_{\tilde{A}-} &= \sigma\{B \cap \{\tilde{A} > t\}, \quad B \in \tilde{\mathcal{F}}_t\} \\
&= \sigma\{B \cap \{\tilde{A} > \tau > t\}, \quad \tau \in T, B \in \tilde{\mathcal{F}}_t\} \\
&= \sigma\{D \cap \{\tilde{A} > t\}, \quad D \in \mathcal{F}_{\tau-}\} \\
&\subset \tilde{\mathcal{G}}_{\tilde{A}-}.
\end{aligned}
$$

Since on the same set $\tilde{\mathcal{F}}_{\tilde{A}-} = \tilde{\mathcal{G}}_{\tilde{A}-} \subset \tilde{\mathcal{G}}_{\tilde{A}} \subset \tilde{\mathcal{F}}_{\tilde{A}}$, it is enough to prove $\tilde{\mathcal{F}}_{\tilde{A}-} = \tilde{\mathcal{F}}_{\tilde{A}}$. To do it we will use a rather unusual feature of our enlargement. Notice

$$
\{\tilde{A} \leq s\} = \{\tilde{A} \leq t\} \cap \{A_t \leq s\} \quad \text{for } s \leq t < t_0.
$$

However $\{A_t \leq s\} \in \mathcal{F}_t$. This implies that the σ-algebra $\mathcal{F}_t$ was augmented only by 1 set. More precisely

$$
\tilde{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(\{\tilde{A} \leq t\}) \ a.s. \tag{1.7}
$$

Let $\mathcal{Z}$ be the set of all non-negative, integrable, $\mathcal{F}_\infty$ measurable random variables, and $Z \in \mathcal{Z}$. Simple algebra and equation (1.7) show that the martingale

$$
\eta_t^Z = E[Z|\tilde{\mathcal{F}}_t] = \xi_1^Z(t) 1_{\{\tilde{A} \leq t\}} + \xi_2^Z(t) 1_{\{\tilde{A} > t\}}, \tag{1.8}
$$

where

$$\xi_1^Z(t) = \frac{E[Z1_{\{\tilde{A}\le t\}}|\mathcal{F}_t]}{P[\tilde{A}\le t|\mathcal{F}_t]}, \quad \text{and} \quad \xi_2^Z(t) = \frac{E[Z1_{\{\tilde{A}>t\}}|\mathcal{F}_t]}{P[\tilde{A}>t|\mathcal{F}_t]}.$$

Observe that the process $E[Z1_{\{\tilde{A}\le t\}}|\mathcal{F}_t]$ is a submartingale and the function $t \to E[Z1_{\{\tilde{A}\le t\}}]$ is continuous, hence there is a *càdlàg* modification of $E[Z1_{\{\tilde{A}\le t\}}|\mathcal{F}_t]$. Thus we can assume without loss of generality that the processes $\xi_1^Z(t)$ and $\xi_2^Z(t)$ are $\mathcal{F}_t$ adapted *càdlàg* processes. This immediately implies that

$$\tilde{\mathcal{F}}_{\tilde{A}} = \sigma(\eta_{\tilde{A}}^Z,\ Z\in\mathcal{Z}) = \sigma(\xi_1^Z(\tilde{A}-),\ Z\in\mathcal{Z}) \subset \tilde{\mathcal{F}}_{\tilde{A}-},$$

since any $\mathcal{F}_t$ adapted *càdlàg* process is *a.s.* continuous at the time $\tilde{A}$.

Lemma 1.2 implies that $\mathcal{F}_t$ is constant on any interval $[s, S_s)$, e.g. for any $t > s$ and $B \in \mathcal{F}_t$ we have $B' \in \mathcal{F}_s$ such that $B \cap \{S_s > t\} = B' \cap \{S_s > t\}$. A similar statement is true for σ-algebra $\mathcal{H}_s$ and consequently[2] for $\tilde{\mathcal{F}}_s$.

To finish the proof we will closely follow the proof that appears in section 2 (page 22) of [29]. Let M_t be any uniformly integrable $\tilde{\mathcal{F}}_t$ martingale such that M_t is 0 on $[0,\tilde{A}]$ and constant on $[t_0,\infty)$. To prove that M_t is 0 on $[0,\infty)$, we define $M_t^s = M_{t\wedge S_s} - M_{t\wedge s}$. Note that $\{S_s < t_0\} \subset \{S_s < \tilde{A}\}$, and therefore the martingale $M_t^s \equiv 0$ on the set $\{S_s < t_0\}$, so $M_t^s = M_t^s 1_{\{t_0\le S_s\}}$. The statement established in the previous paragraph implies that for any $t > s$ there is a $\tilde{\mathcal{F}}_s$ measurable random variable N_t such that $N_t = M_t^s$ on the set $\{s \le t < S_s\}$. Call G a regular version of the law of the pair $(S_s, M_{S_s}^s)$ conditional on $\tilde{\mathcal{F}}_s$, and $G''(t) = G((t,\infty]\times\mathbb{R}\cap[t_0,\infty]\times\mathbb{R}))$.

[2]Using (1.8) we can actually prove that the filtration $\tilde{\mathcal{F}}_s$ is quasi-left-continuous.

If $t \geq s$, we have the following string of $a.s.$ equalities (see [29] for justification):

$$\begin{aligned} N_t G''(t) &= E[N_t 1_{\{t<S_s\}} 1_{\{t_0 \leq S_s\}} \,|\, \tilde{\mathcal{F}}_s] = E[M_t^s 1_{\{t_0 \leq S_s\}} 1_{\{t<S_s\}} \,|\, \tilde{\mathcal{F}}_s] \\ &= E[M_t^s 1_{\{t<S_s\}} \,|\, \tilde{\mathcal{F}}_s] = E[M_{S_s}^s 1_{\{t<S_s\}} \,|\, \tilde{\mathcal{F}}_s] = \int x \, 1_{\{u>t\}} G(du, dx). \quad (1.9) \end{aligned}$$

The functions $G''(t)$ and $\int x \, 1_{\{u>t\}} G(du, dx)$ (taken as a function of t) are $a.s.$ constant on the interval $[0, t_0)$. The fact that the second function is constant follows from $G((t, t_0) \times (\mathbb{R} \setminus \{0\})) = 0$ $a.s.$ To conclude that $M_t^s = 0$ $a.s.$ on the interval $[0, t_0]$ notice that the set $\{G''(0) = 0\}$ is $\mathcal{F}_s$ measurable, and more important $\{G''(0) = 0\} \subset \{S_s < t_0\}$. (The continuity of M^s at the point t_0 is implied by $\tilde{\mathcal{F}}_{t-} \supset \mathcal{F}_{t-} = \mathcal{F}_t = \tilde{\mathcal{F}}_t$.) Since s was arbitrary, we get $M_t = 0$ for $t \in [0, t_0]$. From here we finally obtain $\mathcal{F}_{t_0} = \tilde{\mathcal{F}}_{\tilde{A}} = \tilde{\mathcal{G}}_{\tilde{A}} \subset \mathcal{G}_{t_0}$ on the set $\{\tilde{A} < \infty\} = B_3$. □

The following two examples demonstrate that it is indeed possible to have $P(B_1)$ or $P(B_3)$ bigger then 0. The filtration defined in each example is purely discontinuous. We postpone the proof thereof untill the last chapter. (Example 1.1 is covered by Corollary 4.3 and Example 1.2 is covered by Example 4.3.)

Example 1.1. Let τ_n be sequence of independent random variables with exponential distribution ($\lambda = 1$). Define $\mathcal{F}_t = \sigma\{\{\tau_n \leq s\}; s \leq t\}$ (i.e. $\mathcal{F}_t$ is the filtration generated by the sequence of processes $\{1_{\{\tau_n \leq t\}}\}$). It is easy to see that $\mathcal{F}_t$ is a quasi-left-continuous filtration satisfying the conclusion of Theorem 1.1, and for any $t \in (0, 1)$, $A_t = S_t = t$. Thus $P(B_1) = 1$.

See Figure 1.1 for a simulated sample path of the compensated sum of jumps

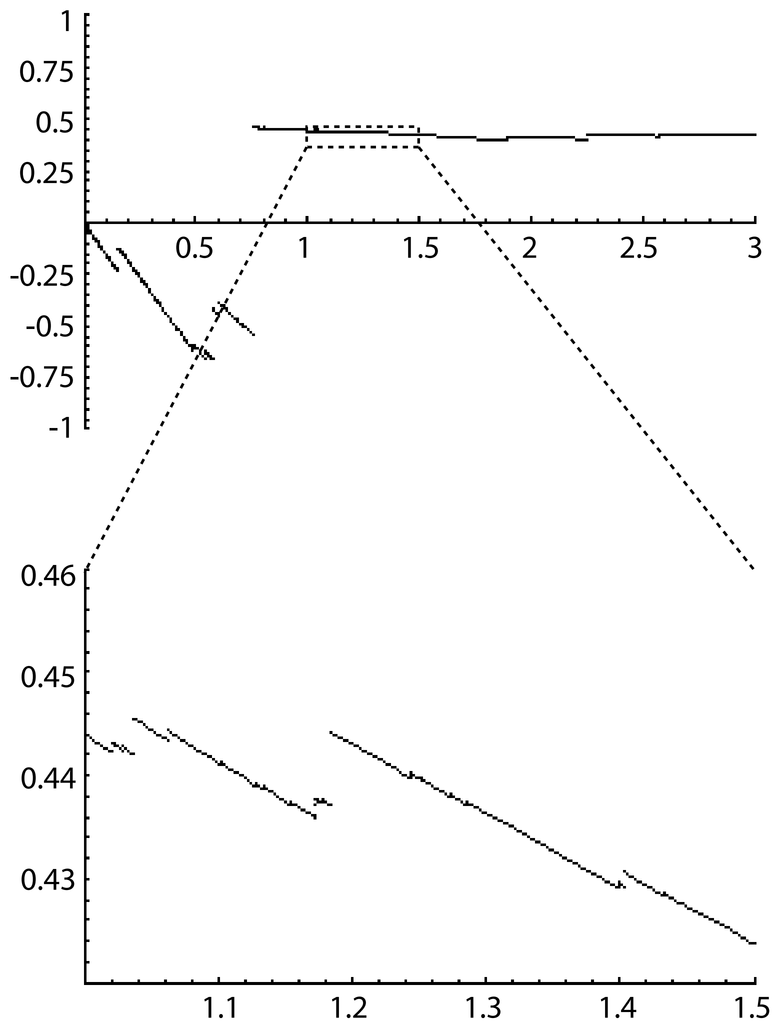

Figure 1.1: Simulated sample path of a martingale with accumulation of jumps everywhere (2 different resolutions).

$\sum_n \frac{1}{n^2}\left(1_{\{\tau_n \le t\}} - C_t^n\right)$. The simulations were performed using MATHEMATICA 4.0 on iMac DVD owned by the author. To draw the picture we used first 1000 terms of the sum.

Example 1.2. Let $B(t)$ be a Brownian motion and $\mathcal{W}_t$ be its natural filtration. Let $Z_t = \inf\{s > t : B(s) = 0\}$, and $D_t = \sup\{s < t : B(s) = 0\}$. Note that $D_t < t < Z_t$ *a.s.*. It follows easily from the strong Markov property of Brownian motion that for any $\mathcal{W}$-stopping time ν we have $P(\nu = D_t) = 0$.

Denote the natural filtration of the process $\mathrm{sign}(B_t)$ by $\mathcal{F}_t$. We will prove in the Example 4.3 that this filtration is purely discontinuous and the set $T = \{Z_q : q \in \mathbb{Q}\}$ is the set of all totally inaccessible $\mathcal{F}_t$-stopping times[3]. Thus $A_t = D_t$ and $P(B_3) = 1$.

[3] In the sense that for any totally inaccessible stopping time τ, $[\tau] \subset \bigcup_{\upsilon \in T} [\upsilon]$.

Chapter 2

Examples

In this chapter we will present several examples that constitute a negative answer to some rather interesting questions. First we will prove that the necessary condition (1.1) in Theorem 1.1 is not a sufficient condition. Then we will go on and prove that a subfiltration of a purely discontinuous filtration does not have to be purely discontinuous.

In the previous chapter we have proved a necessary condition for filtration to be purely discontinuous. However, as the following examples show the condition is not sufficient. Namely we find filtrations that are defined in agreement with the formula (1.1) of the main theorem, but are not purely discontinuous.

Example 2.1. Let $\{\Pi_t^n\}$ be a sequence of independent Poisson processes with intensities $\lambda_n = n$, W_t be a Brownian motion independent of $\{\Pi_t^n\}$, and $0 < g < 1$ be an increasing, continuously differentiable function. Denote the k-th jump of Π^n by τ_k^n.

Define $\mathcal{F}_t$ as the smallest σ-algebra for which the processes $\{g(W_{\tau_k^n})1_{\{\tau_k^n \leq t\}}\}$ are all adapted. Then $\mathcal{F}_t$ satisfies the conclusion of Theorem 1.1 with $T = \{\tau_k^n,\ n, k \in \mathbb{N}\}$, and W_t is an adapted continuous martingale.

Proof. Since the Poisson processes involved are mutually independent the stopping times in T have disjoint graphs. It is a well-known fact that times of jumps of Poisson process are totally inaccessible with respect to its natural filtration. Since the processes Π^n and W are mutually independent, the compensator of τ_k^n calculated under the natural filtration of Π^n is equal to a compensator calculated under $\mathcal{F}_t$. Hence the compensator is continuous and the stopping time τ_k^n is totally inaccessible with respect to the filtration $\mathcal{F}_t$.

Define $\tau^n = \sum_{k=1}^{\infty} \tau_k^n 1_{\{\tau_k^n \le t < \tau_{k+1}^n\}}$. Then τ_n and $g(W_{\tau^n})$ are $\mathcal{F}_t$-measurable random variables. We need to prove that $\tau^n \to t$ *a.s.* Calculate

$$\{t - \tau^n > \varepsilon\} \subset \bigcup_{k=0}^{m-1} \{\tau_{k+1}^n - \tau_k^n > \varepsilon\} \cup \{\tau_m^n < t\},$$

$$P(\tau_m^n < t) \le P(|\tau_m^n - \frac{m}{n}| > \frac{m}{n} - t) \le \frac{\frac{m}{n^2}}{(\frac{m}{n} - t)^2} = \frac{m}{(m - nt)^2},$$

for $m/n > t$ by Chebyshev's inequality. Choose $m = n^2$ and calculate for $n > t$

$$P(t - \tau^n > \varepsilon) \le m e^{-n\varepsilon} + \frac{m}{(m - nt)^2} = n^2 e^{-n\varepsilon} + \frac{1}{(n - t)^2}. \tag{2.1}$$

Since the right-hand side of (2.1) is summable, we conclude by Borel-Cantelli's lemma that $P(t - \tau^n > \varepsilon,\ \text{i. o.}) = 0$. Hence $\tau^n \to t$ *a.s.*, and consequently $g(W_{\tau^n}) \to g(W_t)$ *a.s.*. Thus W_t is an $\mathcal{F}_t$ adapted process. Finally the independence of Π^n and W ensures that W is a Brownian motion with respect to the filtration $\mathcal{F}_t$. □

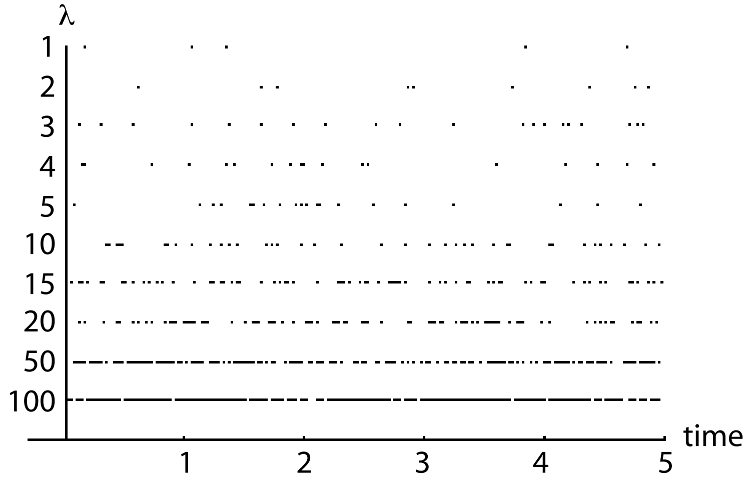

Figure 2.1: Simulated times of jumps for Poisson processes with different λ.

We will prove in the last chapter that the σ-algebra generated by the sequence of independent Poisson processes is purely discontinuous (see Theorem 4.2). That means that we "smuggled in" the continuous martingale into the filtration using the size of jumps. A natural question arises whether filtrations generated by jumps only are purely discontinuous.

The next example shows that there is a filtration generated by jumps of size 1 that still supports continuous martingale.

Example 2.2. Let U_k be a sequence of independent random variables with uniform distribution on $(0,1)$. Let W be a Brownian motion independent of the sequence $\{U_k\}$. Denote $\mathcal{W}_t$ the natural filtration of W. There is a strictly increasing continuous process G_t, $0 \leq G \leq 1$ that generated $\mathcal{W}_t$. Define[1]

$$\tau_k = G^{-1}_{U_k(\omega)}(\omega), \tag{2.2}$$

[1]The symbol $G^{-1}_{\bullet}(\omega)$ is the inverse function to the function $G_{\bullet}(\omega)$. If $U_k(\omega) \geq G_{\infty}(\omega)$, then $\tau_k(\omega) = \infty$.

and

$$\mathcal{F}_t = \sigma\{\{\tau_k \le s\} : \quad s \le t,\ k \in \mathbb{N}\}. \tag{2.3}$$

Then $\mathcal{F}_t$ satisfies the conclusion of Theorem 1.1, but W_t is an adapted continuous martingale.

Proof. First we find the suitable G. Let

$$G_t(\omega) = \int_0^t e^{-s} g(W_s(\omega)) ds,$$

where $g(t) = \frac{1}{\pi}\arctan(t) + \frac{1}{2}$. This process is an increasing, $\mathcal{W}_t$ adapted, continuous process, and $0 < G_t < 1$ *a.s.* The continuity of the entities involved implies

$$\frac{G_t - G_{t-\varepsilon}}{\varepsilon} = \frac{1}{\varepsilon}\int_{t-\varepsilon}^t e^{-s} g(W_s) \to e^{-t} g(W_t).$$

Hence W is adapted to the natural filtration of G.

It follows from formula (2.2) that $\{\tau_k\}$ is a stationary sequence of random variables and $\mathcal{W}_\infty$ is its shift invariant σ-algebra. Moreover

$$P[\tau_k \le t \,|\, \mathcal{W}_\infty] = P[U \le G_t \,|\, \mathcal{W}_\infty] = G_t. \tag{2.4}$$

Thus the well-known ergodic theorem (see Theorem 0.17) implies

$$\frac{1}{n}\sum 1_{\{\tau_k \le t\}} \to E[1_{\{\tau_1 \le t\}} \,|\, \mathcal{W}_\infty] = G_t \ a.s.$$

We proved that W_t is $\mathcal{F}_t$ adapted. However, we still have to check that W_t is a martingale with respect to this bigger σ-algebra.

Fix $t < s$ and denote $\tilde{G}_s = G_{s \wedge t}$. Then $\tau_k \wedge t = G_t^{-1}(U_k) \wedge t = \tilde{G}_t^{-1}(U_k) \wedge t$. However $\tilde{G}$ and $\tilde{G}^{-1}$ are $\mathcal{W}_t \times \mathcal{B}[0,t]$ measurable, whence $\tau_k \wedge t$ is independent of $W_s - W_t$. Thus W is a Brownian motion under the filtration $\mathcal{F}$.

To finish the proof notice that the random variables τ_k have continuous distribution, $\tau_1 \amalg_{\mathcal{W}_\infty} f(\tau_2, \tau_3, \dots)$ for any measurable function f, and deduce that τ_k are totally inaccessible stopping times. $\square$

The following simple example shows that a subfiltration of a purely discontinuous filtration is not necessarily purely discontinuous.

Example 2.3. Let W_t be a Brownian motion and τ be a random variable uniformly distributed on $(0,1)$ independent of W. Denote $\mathcal{H}_t$ the smallest filtration that makes both W_t and $1_{\{\tau \le t\}}$ adapted. Define filtration

$$\mathcal{F}_t = \sigma\{A \cap \{\tau \le t\} \; : \; A \in \mathcal{H}_1\} \tag{2.5}$$

and a process $X_t = W_{1 \wedge t} - W_{\tau \vee (1 \wedge t)}$. Let $\mathcal{G}_t$ denote the natural filtration of X.

Then $\mathcal{F}_t$ is a purely discontinuous filtration, X is a continuous $\mathcal{G}_t$-martingale, and $\mathcal{G}_t \subset \mathcal{F}_t$ for each t.

Proof. The filtration $\mathcal{F}_t$ was generated by one jump. It is a purely discontinuous filtration by Theorem 0.12. The process X is a continuous martingale with respect to both $\mathcal{H}_t$ and $\mathcal{G}_t$ by Proposition 0.1. Finally, $\mathcal{G}_t \subset \mathcal{F}_t$, because X_t is $\mathcal{H}_1$ measurable for any $t \geq 0$, and $X_t = 0$ for $t \leq \tau$. $\square$

Notice that X_t is an $\mathcal{F}_t$-adapted process, but it is not an $\mathcal{F}_t$-martingale. This was accomplished by "moving the information" in the filtration $\mathcal{H}_t$ forward.

Chapter 3

Necessary and Sufficient Condition

In this chapter we prove a general necessary and sufficient condition for a filtration to be purely discontinuous. Our condition is similar to the condition discovered by J. Jacod and M. Yor (1977), [31], that claims that a martingale M has a strong representation property if and only if the original probability is an extreme point in the set of all probability measures that keep M a martingale. The big drawback of our condition is that it is usually very hard to verify.

Denote $C_t^{\tau,\xi}$ the compensator of $\xi 1_{\{\tau \leq t\}}$, where τ is a totally inaccessible stopping time and ξ is a bounded $\mathcal{F}_\tau$ measurable random variable.

Theorem 3.1. *Let $\mathcal{F}_t$ be a quasi-left-continuous filtration. Put*

$$\Gamma = \left\{ P' : \begin{array}{c} P' \textit{ is a probability on } \mathcal{F}_\infty \textit{ such that} \\ C_t^{\tau,\xi} \textit{ is a compensator of } \xi 1_{\{\tau \leq t\}} \textit{ under } P' \end{array} \right\}.$$

Then $\mathcal{F}$ is purely discontinuous and $\mathcal{F}_0$ is a trivial σ-algebra if and only if P is an *extreme point of* Γ.

The proof is a an adaptation of the proof of a result similar to the theorem mentioned in the first paragraph of this chapter (Theorem 13.21 from [16]).

Proof. Assume that P is an extreme point of Γ. Let ζ be bounded $\mathcal{F}_0$ measurable random variable with $E\zeta = 0$ and N be a bounded continuous martingale with $N_0 = 0$. Denote $M = \zeta + N$. Assume without loss of generality that

$$|M| \leq 1.$$

Define

$$dP_1 = \left(1 + \frac{M_\infty}{2}\right), \quad dP_2 = \left(1 - \frac{M_\infty}{2}\right).$$

Since $EM_0 = 0$, P_1, P_2 are probability measures equivalent to P and

$$P = \frac{1}{2}P_1 + \frac{1}{2}P_2.$$

For arbitrary but fixed τ and ξ denote $X_t = \xi 1_{\{\tau \leq t\}} - C_t^{\tau,\xi}$. The Girsanov's Theorem (see Theorem 0.16) implies that the P semimartingale

$$X' = X - \frac{1}{Z_-} \cdot \langle Z, X \rangle, \tag{3.1}$$

where $Z = (1 + M_\infty/2)$, is a P_1 martingale. However, $\langle Z, X \rangle = 0$ and therefore $P_1 \in \Gamma$. Hence $P_1 = P$ and subsequently $M \equiv 0$.

Assume that P is not an extreme point of Γ, namely $P = \lambda P_1 + (1 - \lambda)P_2$ for some suitable $P_1, P_2 \in \Gamma$. Clearly $P_1 << P$. Define

$$Z = \frac{dP_1}{dP}, \quad Z_t = E[Z \mid \mathcal{F}_t].$$

If $\Delta Z \neq 0$ we can find a totally inaccessible stopping time $\tau \neq \infty$ such that $[\tau] \subset [\Delta Z \neq 0]$, and ΔZ_τ is bounded. The process $X_t = \Delta Z_\tau 1_{\{\tau \le t\}} - C_t^{\tau, \Delta Z_\tau}$ is a martingale and the Girsanov's theorem implies that

$$X' = X - \frac{1}{Z_-} \cdot \langle Z, X \rangle$$

is a P_1 martingale. However the predictable process of bounded variation $\frac{1}{Z_-} \cdot \langle Z, X \rangle$ is not a constant P_1 $a.s.$, and therefore the compensator of $\Delta Z_\tau 1_{\{\tau \le t\}}$ under P_1 is not $C_t^{\tau, \Delta Z_\tau}$. Thus $\Delta Z \equiv 0$. $\square$

Notice that the proof really used only probabilities absolutely continuous with respect to P. Hence we can reformulate the theorem in the following way.

Corollary 3.2. *Let $\mathcal{F}_t$ be a quasi-left-continuous filtration. Put*

$$\Gamma' = \left\{ P' : \begin{array}{l} P' << P \textit{ is a probability on } \mathcal{F}_\infty \textit{ such that} \\ C_t^{\tau, \xi} \textit{ is a compensator of } \xi 1_{\{\tau \le t\}} \textit{ under } P' \end{array} \right\}.$$

Then $\mathcal{F}$ is purely discontinuous if and only if $\Gamma' = \{P\}$.

Remark 3.1. As noted earlier the condition of Theorem 3.1 is rather difficult to verify. However it can be verified in the case of σ-algebra generated by a purely discontinuous Levy process. It is a known fact that the probability measure is uniquely determined by the predictable characteristics of the Levy process and all the compensators are uniquely determined by the Levy measure.

Chapter 4

Sufficient Conditions

In this chapter we will investigate various sufficient conditions for purely discontinuous filtrations as well as some auxiliary results. These conditions will be easier to verify than the condition of Theorem 3.1.

We will refer to the following sets of assumptions corresponding to the conclusion of Theorem 1.1.

Assumptions.

A1. Let $\tau_n \geq 0$ be a sequence of random variables with disjoint graphs, ξ_n be a sequence of random variables, and $\mathcal{F}_t$ be the filtration generated by processes $\{\xi_n 1_{\{\tau_n \leq t\}}\}$.

A2. We assume A1 and $\xi_n = 1$.

R1. In addition to A1 we assume that the filtration $\mathcal{F}_t$ is quasi-left-continuous, the stopping times τ_n that generated the filtration are all totally inaccessible, and if τ' is a totally inaccessible stopping time, then

$$[\tau'] \subset \bigcup_n [\tau_n]. \tag{4.1}$$

R2. We assume R1 and $\xi_n = 1$.

Clearly, if a filtration satisfies assumption A2 (resp. R2) it also satisfies assumption A1 (resp. R1). We will also use the following notation.

Definition 4.1. Let filtration $\mathcal{F}_t$ satisfy Assumption A1 we define $\mathcal{F}_t^k$ as filtration defined by processes $\{\xi_n 1_{\{\tau_n \le t\}},\ n \le k\}$, and $\hat{\mathcal{F}}^k$ as filtration defined by processes $\{\xi_n 1_{\{\tau_n \le t\}},\ n > k\}$.

We say that the filtration $\mathcal{F}_t$ is tail-free if the σ-algebra

$$\hat{\mathcal{F}}_t^\infty = \bigcap_{k \in \mathbb{N}} \hat{\mathcal{F}}_t^k$$

is trivial for $t = \infty$.

The following examples show that in the assumption R1 the quasi-left-continuity and the relation (4.1) does not hold automatically and should be assumed.

Example 4.1. Let W be a Brownian motion starting at 1. Define

$$Z_q = \inf\{t : \quad W_t = 0,\ t > q\}.$$

It is a well-known fact that $P[q < Z_q < \infty] = 1$. Let $\{q_n\}_{n=1}^\infty = \mathbb{Q} \cap (0, \infty),\ q_0 = 0$.

Since the random variables $\{Z_q\}$ do not have disjoint graphs, define

$$\tilde{Z}_{g_n} = \begin{cases} \infty & \text{on} \quad A = \bigcup_{k=0}^{n-1} \{Z_{q_n} = Z_{q_k}\}, \\ Z_{q_n} & \text{on} \quad \Omega \setminus A. \end{cases}$$

Denote by $\mathcal{F}_t$ the filtration generated by the processes $\{1_{\{\tilde{Z}_q \le t\}}\}$. Then the properties of Brownian motion assure that Z_q are totally inaccessible stopping times. We have defined $\tilde{Z}_q$ in such a way that $\tilde{Z}_q = Z_0$ for any rational $q > 0$. Furthermore Z_0 was not involved in the definition of the filtration $\mathcal{F}_t$. However,

$$Z_0 = \inf_{q \in \mathbb{Q} \cap (0,1)} \tilde{Z}_q$$

is a totally inaccessible stopping time with respect to the filtration $\mathcal{F}_t$.

Example 4.2. Let $\{e_k\}$ be a sequence of i.i.d. exponential ($\lambda = 1$) variables defined on a probability space $(\Omega_1, \mathcal{F}_1, P_1)$. Let $(\Omega_2 = \{0, 1\}, \mathcal{F}_2 = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}, P_2 = (1/2, 1/2))$ be another probability space. Define the probability space $(\Omega, \mathcal{F}, P)$ as the product space of Ω_1 and Ω_2. Also define random variables

$$\tau_k(\omega_1, \omega_2) = \begin{cases} \infty & \text{for } \omega_2 = 1, \\ e_k(\omega_1) + 1 & \text{for } \omega_2 = 0, \end{cases} \tag{4.2}$$

and filtration $\mathcal{F}_t = \sigma\{\{\tau_k \le s\} : \; s \le t, \; k \in \mathbb{N}\}$. Clearly the filtration $\mathcal{F}_t$ is of the form of Assumption R2, perhaps short of being quasi-left-continuous. We will indeed prove it is not quasi-left-continuous.

Notice that the σ-algebra $\mathcal{F}_s$ is trivial for $s < 1$. Hence $\mathcal{F}_{1-}$ is also trivial.

However, the right-continuity of the filtration implies that the set

$$A = \bigcap_{1<s} \bigcup_{k\in\mathbb{N}} \{\tau_k \le s\} = \Omega_1 \times \{0\}$$

is $\mathcal{F}_1$ measurable, and $P(A) = 1/2$. Thus $\mathcal{F}_{1-} \neq \mathcal{F}_1$.

Remark 4.1. In the previous example we have used the fact that $S_1 = 1$. It is not difficult to prove that if a filtration satisfies Assumption R1 minus the quasi-left-continuity and for a predictable stopping time τ the stopping time $S_\tau > \tau$ *a.s.*, then $\mathcal{F}_{\tau-} = \mathcal{F}_\tau$.

We now proceed to the first theorem of this section.

Theorem 4.1. *Let filtration $\mathcal{F}_t$ satisfy assumption A1. Additionally, for each k there is n_k, such that for any $\xi \in \mathcal{F}^k_\infty$,*

$$E[\xi \,|\, \mathcal{F}_t] = E[\xi \,|\, \mathcal{F}^{n_k}_t]. \tag{4.3}$$

Then the filtration $\mathcal{F}_t$ is purely discontinuous.

We say that a filtration is quasi-Markov if it satisfies the assumptions of Theorem 4.1.

Proof. Let ξ_t be a continuous, square integrable $\mathcal{F}_t$-martingale satisfying $\xi_0 = 0$. There is a sequence ξ^k of square integrable $\mathcal{F}^k_\infty$-measurable random variables such that

$$\xi^k \xrightarrow{L^2} \xi_\infty. \tag{4.4}$$

Define $\xi_t^k = E[\xi^n \,|\, \mathcal{F}_t] = E[\xi^{n_k} \,|\, \mathcal{F}_t]$. The last equality is the assumption of the theorem.

The filtration $\mathcal{F}_t^n$ is generated by a finite number of stopping times and it is jumping. The Theorem 0.12 implies that ξ_t^k is a compensated sum of jumps, whence

$$\langle \xi^n, \xi \rangle = 0. \tag{4.5}$$

This and equation 4.4 imply $\xi = 0$ $a.s.$ $\square$

The Assumption 1 in Theorem 4.1 was used only to establish the fact that the filtrations $\mathcal{F}_t^n$ are purely discontinuous. Thus we have already proved the following theorem.

Theorem 4.2. *Let $\mathcal{F}_t^n$ be an increasing sequence of purely discontinuous filtrations. And $\mathcal{F} = \bigvee \mathcal{F}^n$. Assume that for each k there is n_k, such that for any $\xi \in \mathcal{F}_\infty^k$,*

$$E[\xi \,|\, \mathcal{F}_t] = E[\xi \,|\, \mathcal{F}_t^{n_k}]. \tag{4.6}$$

Then the filtration $\mathcal{F}$ is purely discontinuous.

We will now state two interesting examples.

Corollary 4.3. *Let $\tau_n \geq 0$ be a sequence of independent, non-atomic random variables, and $\mathcal{F}_t$ be the filtration generated by the processes $\{\tau_n \leq t\}$. Then*

1. *τ_n are totally inaccessible stopping times with respect to $\mathcal{F}_t$.*

2. *Any $\mathcal{F}_t$ adapted continuous martingale is constant a.s.*

Proof. To prove statement 1 it is enough to prove that the compensator of $1_{\{\tau_n \leq t\}}$ is continuous. Define $\mathcal{F}_t^{(k)}$ as filtration defined by process $\xi_k 1_{\{\tau_k \leq t\}}$, and $\hat{\mathcal{F}}^{(k)}$ as filtration defined by processes $\{\xi_n 1_{\{\tau_n \leq t\}},\ n \neq k\}$. Calculate

$$E 1_{\{s<\tau_n \leq t\}} 1_{A \cap B} = P(B) E 1_{\{s<\tau_n \leq t\}} 1_A,$$

for $A \in \mathcal{F}_s^{(n)}$ and $B \in \hat{\mathcal{F}}_s^{(n)}$. Thus

$$E[1_{\{s<\tau_n \leq t\}} \,|\, \mathcal{F}_s] = E[1_{\{s<\tau_n \leq t\}} \,|\, \mathcal{F}_s^{(n)}]. \tag{4.7}$$

Therefore the compensator of $1_{\{\tau_n \leq t\}}$ is the same in both filtrations $\mathcal{F}_t$ and $\mathcal{F}_t^{(n)}$, hence it is continuous (see Proposition 0.13).

We now proceed to the proof of the part 2. Let ξ be a bounded $\mathcal{F}_\infty^n$ measurable random variable. Clearly $\xi = f(\tau_1, \ldots, \tau_n)$, for a suitable measurable function f. For $s \in \mathbb{R}$, $A \in \mathcal{F}_s^n$, and $B \in \hat{\mathcal{F}}_s^n$ calculate

$$E\,\xi 1_{A\cap B} = P(B) E\,\xi 1_A = P(B) E[E[\xi\,|\mathcal{F}_s^n] 1_A] = E[E[\xi\,|\mathcal{F}_s^n] 1_{A \cap B}].$$

Monotone class argument implies

$$M_t = E[\xi \mid \mathcal{F}_t] = E[\xi \mid \mathcal{F}_t^n]. \tag{4.8}$$

This establishes the quasi-Markov property and the corollary follows. □

The next example is very important. It deals with filtration generated by a Levy process. The literature of Levy processes is very well developed (see [3], [16], [30]). Natural filtration $\mathcal{F}_t$ of a Levy process X has been proved to be quasi-left-continuous([16] Theorem 13.47). In fact the very notion of quasi-left-continuity has been inspired by Levy processes. Another well-known fact is that X has a week representation property with respect to $\mathcal{F}_t$ ([16] Theorem 13.48). The Corollary 4.4 is an immediate consequence of this fact. However, we have decided to present a proof that does not use this fact.

Corollary 4.4. *Let X be a purely discontinuous process with independent increments without fixed discontinuities (sometimes called a Levy process) and $\mathcal{F}_t$ be its natural filtration. Then any $\mathcal{F}_t$ adapted local continuous martingale is constant a.s.*

Proof. We can assume without loss of generality that X is a martingale and $\Delta X \in [-1, 1]$. Define $X^{(k)}$ to be a compensated sum of jumps of X whose size belongs to $[-1/k, -1/(k+1)) \cap (1/(k+1), 1/k]$. The processes $X^{(k)}$ are mutually independent processes with independent increments (see [34] Lemma 13.6).

Define $\mathcal{F}_t^{(i)}$ as the natural filtration of $X^{(i)}$, $\mathcal{F}^n = \bigvee_{i=1}^n \mathcal{F}^{(i)}$, and $\hat{\mathcal{F}}^n = \bigvee_{i=n+1}^\infty \mathcal{F}^{(i)}$. The fact that the $\mathcal{F}_t$ is a natural filtration of X and the relation $X = \sum X^{(i)}$ imply that $\mathcal{F} = \bigvee \mathcal{F}^{(i)}$. Notice that the filtration $\mathcal{F}^n$ is jumping and therefore purely discontinuous by the Theorem 0.12.

Since the σ-algebras $\mathcal{F}^n_\infty$ and $\hat{\mathcal{F}}^n_\infty$ are independent, we can calculate

$$\zeta^n_t = E[\zeta^n \,|\, \mathcal{F}_t] = E[\zeta^n \,|\, \mathcal{F}^n_t], \tag{4.9}$$

for any integrable $\mathcal{F}^n_\infty$-measurable random variable ζ^n. The rest of the proof follows from Theorem 4.2. $\square$

Before proceeding to the last example of this section we need to state one well-known result.

Lemma 4.5. *Let X be a purely discontinuous martingale and $\mathcal{F}_t$ be its natural filtration. Then if X has a weak representation property then $\mathcal{F}_t$ is purely discontinuous.*

Recall that X has a weak representation property if for any martingale M_t with $M_0 = 0$, there are two suitable predictable processes[1] $H_1,\ H_2$ such that

$$M^c_t = \int_0 H_1 dX^c, \quad M^d_t = \int_{[0,t]\times\mathbb{R}} H_2(x,s)\tilde{\mu}(ds,dx),$$

where M^c (resp. M^d) are the continuous (resp. purely discontinuous parts) of M, X^c is the continuous part of X and $\tilde{\mu}$ is the compensated jump measure of X. Since $X^c = 0$ the lemma is just a simple consequence of the definition.

Example 4.3. Let B be a Brownian motion and $\mathcal{F}$ be the natural filtration of the process $\mathrm{sign}(B_t)$. Then $\mathcal{F}_t$ is purely discontinuous.

[1] The second process is defined on $\mathbb{R}\times\mathbb{R}_+\times\Omega$ and is $\mathcal{B}\times\mathcal{P}$ measurable.

Proof. Denote $M_t = E[B_t \,|\, \mathcal{F}_t]$. M_t is called the Azéma Martingale. The following properties of the Azéma martingale are known: (For a reference see [38], section IV.6.)

- If $A_t = \sup\{s \le t : \; B_s = 0\}$ then

$$M_t = \operatorname{sign}(B_t)\sqrt{\frac{\pi}{2}(t - A_t)}. \tag{4.10}$$

- M_t is a purely discontinuous martingale with respect to $\mathcal{F}_t$.

- The natural filtration of M_t is $\mathcal{F}_t$.

- The square characteristics $\langle M \rangle_t = \frac{2}{\sqrt{\pi}} t$.

For simulated[2] sample path of the Azéma Martingale see Figure 4.1.

M. Emery in [15] (page 79) proved that M_t satisfies the chaotic decomposition property that is a special case of strong representation property. Since strong representation property implies weak representation property (take $H_2(t,x) = H_t x$), we conclude that $\mathcal{F}_t$ is purely discontinuous. Note that the set of possible jumps T of the Theorem 1.1 corresponds to the 0 of the Brownian motion B. $\square$

We conclude this chapter with results on change of the probability measure.

Lemma 4.6. *Let $(\Omega, \mathcal{F}_\infty, P)$ be a probability space and $P' \sim P$. Then the filtration $\mathcal{F}_t$ is purely discontinuous under P' if and only if it is purely discontinuous under P.*

Proof. Let X be a continuous martingale under P. Denote

$$Z_t = E[\frac{dP'}{dP} \,|\, \mathcal{F}_t] \tag{4.11}$$

[2] The simulations were performed using MATHEMATICA 4.0 on iMac DVD owned by the author.

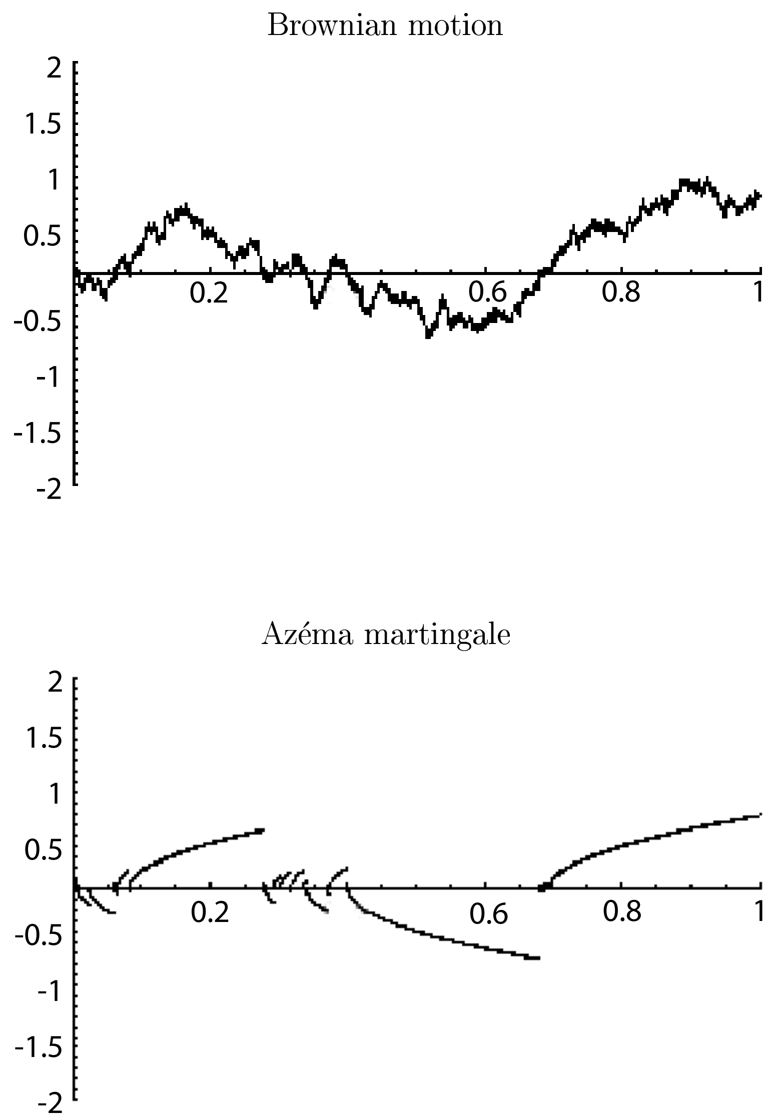

Figure 4.1: Simulated sample paths of Brownian motion and the associated Azéma martingale.

(The calculation in (4.11) is done under P.) Girsanov's theorem (see Theorem 0.16) implies that the process

$$X' = X - \frac{1}{Z_-} \cdot \langle X, Z \rangle \tag{4.12}$$

is a local martingale under P' and $\langle X \rangle_P = \langle X' \rangle_{P'}$. The process $\frac{1}{Z_-} \cdot \langle X, Z \rangle$ is a continuous process of bounded variation. Thus X' is a non-constant continuous local martingale under P'. The statement of the lemma follows. $\square$

Remark 4.2. There is no disagreement with Theorem 3.1, because compensators calculated under P are not necessarily equal to the compensators calculated under P'.

We will now investigate a particular change of measure. We need to do a few calculations and introduce some notation prior to stating next theorem.

Consider the assumption R2. Assume that for any $l \leq k$ the distribution of $\tau_l, \ldots, \tau_k$ given $\hat{\mathcal{F}}^k_\infty$ is equivalent to the the measure $P^{l,k}$, which is the unconditional distribution of $\tau_l, \ldots, \tau_k$, with the density bounded away from 0 and ∞. Formally we assume the following: For any measurable bounded f,

$$\begin{aligned} &E[f(\tau_l, \ldots, \tau_k, \tau_{k+1}, \ldots) \,|\, \hat{\mathcal{F}}^k_\infty] \\ &\qquad = \int f(\tau_l, \ldots, \tau_k, \tau_{k+1}, \ldots) \rho^{l,k}(\tau_l, \ldots, \tau_k, \tau_{k+1}, \ldots) P^{l,k}(d\tau_l, \ldots, d\tau_k), \end{aligned} \tag{4.13}$$

where $0 < c_{l,k} < \rho^{l,k} < C_{l,k} < \infty$ is a measurable function such that

$$\int \rho^{l,k}(\tau_l, \ldots, \tau_k, \tau_{k+1}, \ldots) P^{l,k}(d\tau_l, \ldots, d\tau_k) = 1.$$

For $n \geq k$ denote $\tilde{\rho}_n^{l,k} = E[\rho^{l,k} \,|\, \mathcal{F}_\infty^n]$. Since $\tilde{\rho}_n^{l,k}$ is a positive discrete time martingale, the martingale convergence theorem implies that

$$\tilde{\rho}_n^{l,k} \to \rho^{l,k} \; a.s. \tag{4.14}$$

We can choose versions of the conditional expectations in the definition of $\tilde{\rho}_n^{l,k}$, so that the convergence in (4.14) is everywhere. Define

$$r_n^{l,k}(\tau_{k+1}, \ldots, \tau_n) = \int \tilde{\rho}_n^{l,k}(\tau_l, \ldots, \tau_n) P^{l,k}(d\tau_l, \ldots, d\tau_k) \quad \text{and} \quad \rho_n^{l,k} = \frac{\tilde{\rho}_n^{l,k}}{r_n^{l,k}}.$$

Since $0 < \tilde{\rho}_n^{l,k} < C_{l,k}$, the dominated convergence theorem implies that

$$\lim_{n\to\infty} r_n^{l,k} = \int \rho^{l,k}(\tau_l, \ldots, \tau_k, \tau_{k+1}, \ldots) P^{l,k}(d\tau_l, \ldots, d\tau_k) = 1.$$

Hence $\rho_n^{l,k} \to \rho^{l,k}$ and therefore

$$\lim_{n\to\infty} \frac{\rho^{l,k}}{\rho_n^{l,k}} = 1 \; a.s. \tag{4.15}$$

Let $\varepsilon_k \downarrow 0$ be a summable sequence. Choose a sequence n_k in such a way that $n_0 = 1$ and if $k > 1$

$$P\left[\left|\frac{\rho^{n_{k-1},n_k}}{\rho_{n_{k+1}}^{n_{k-1},n_k}} - 1\right| > \varepsilon_k\right] < \varepsilon_k. \tag{4.16}$$

Borel-Cantelli's lemma implies that

$$P\left[\left|\frac{\rho^{n_{k-1},n_k}}{\rho^{n_{k-1},n_k}_{n_{k+1}}}-1\right|>\varepsilon_k \text{ i.o.}\right]=0.$$

Thus

$$\sum_{k=1}^{\infty}\left|\frac{\rho^{n_{k-1},n_k}}{\rho^{n_{k-1},n_k}_{n_{k+1}}}-1\right|<C(\omega)<\infty,$$

whence

$$\rho=\prod_{k=1}^{\infty}\frac{\rho^{n_{k-1},n_k}}{\rho^{n_{k-1},n_k}_{n_{k+1}}} \quad \text{converges } a.s. \tag{4.17}$$

The theory of infinite products assures that we can find a modification of ρ such that $0<\rho<\infty$.

To simplify the notation we will use the following relabeling:

$$\vec{\tau}_k=(\tau_{n_{k-1}},\ldots,\tau_{n_k}),$$

$$P^k(d\vec{\tau}_k)=P^{n_{k-1},n_k}(d\tau_{n_{k-1}},\ldots,d\tau_{n_k}),$$

$$\mathcal{F}^k=\mathcal{F}^{n_k}_{\infty}, \quad \hat{\mathcal{F}}^k=\hat{\mathcal{F}}^{n_k}_{\infty}, \quad \hat{\mathcal{F}}^{\infty}=\hat{\mathcal{F}}^{\infty}_{\infty},$$

$$\rho_k=\rho^{n_{k-1},n_k}, \quad \hat{\rho}_k=\rho^{n_{k-1},n_k}_{n_{k+1}}.$$

Using the new notation the relation (4.13) transcribes to

$$E[f(\vec{\tau}_k, \vec{\tau}_{k+1}, \dots) \,|\, \hat{\mathcal{F}}^k] = \int f(\vec{\tau}_k, \vec{\tau}_{k+1}, \dots)\rho_k(\vec{\tau}_k, \vec{\tau}_{k+1}, \dots)P^k(d\vec{\tau}_k). \tag{4.18}$$

Similarly $\rho = \prod_{k=1}^{\infty} \frac{\rho_k(\vec{\tau}_k, \vec{\tau}_{k+1}, \dots)}{\hat{\rho}_k(\vec{\tau}_k, \vec{\tau}_{k+1})}$.

We want to change a probability measure in such a way that under the new measure the conditional distribution of $\vec{\tau}_k$ given $\hat{\mathcal{F}}^k$ depends only on $\vec{\tau}_{k+1}$. Let us state the following theorem.

Theorem 4.7. *Assume R2, equations (4.13), (4.17), and use the notation above.*

Suppose there is a probability measure P', such that for any bounded measurable f and for any k

$$E'[f(\vec{\tau}_k, \vec{\tau}_{k+1}, \dots) \,|\, \hat{\mathcal{F}}^k] = \int f(\vec{\tau}_k, \vec{\tau}_{k+1}, \dots)\hat{\rho}_k(\vec{\tau}_k, \vec{\tau}_{k+1})P^k(d\vec{\tau}_k), \tag{4.19}$$

and $\hat{\mathcal{F}}^\infty$ is a trivial σ-algebra under both P and P'. Then $P \sim P'$ and $dP = \rho \cdot dP'$.

If $dP' = \frac{1}{\rho} \cdot dP$ is a probability measure (it is automatically equivalent to P since $0 < \rho < \infty$), then the formula (4.19) is valid.

We will first prove the following version of conditional Fatou lemma.

Lemma 4.8. *Suppose $X_n \geq 0$, $Y = \liminf X_n$ is integrable, and $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \dots$ is a decreasing sequence of σ-algebras. Denote $\mathcal{F}_\infty = \bigcap_n \mathcal{F}_n$. Then*

$$\liminf_{n\to\infty} E[X_n \,|\, \mathcal{F}_n] \geq E[\liminf_{n\to\infty} X_n \,|\, \mathcal{F}_\infty] \text{ a.s.}$$

Proof. Let $Y_n = \inf_{k \geq n} X_k$. Since $0 \leq Y_n \leq X_n$, Y_n is integrable and $Y_n \uparrow Y$. For $Z_n = Y - Y_n$ and $n \geq k$, $E[Z_n \,|\, \mathcal{F}_n] \leq E[Z_k \,|\, \mathcal{F}_n]$. Hence

$$0 \leq \limsup_{n\to\infty} E[Z_n \,|\, \mathcal{F}_n] \leq E[Z_k \,|\, \mathcal{F}_\infty] \to E[0 \,|\, \mathcal{F}_\infty] \ a.s.$$

However

$$0 = \limsup_{n\to\infty} E[Z_n \,|\, \mathcal{F}_n] = E[Y \,|\, \mathcal{F}_\infty] - \liminf_{n\to\infty} E[Y_n \,|\, \mathcal{F}_n] \ a.s.,$$

whence

$$\liminf_{n\to\infty} E[X_n \,|\, \mathcal{F}_n] \geq \liminf_{n\to\infty} E[Y_n \,|\, \mathcal{F}_n] = E[\liminf_{n\to\infty} X_n \,|\, \mathcal{F}_\infty] \ a.s.$$

□

Proof of Theorem 4.7. Let $\xi_1 = f_1(\vec{\tau}_1), \ldots, \xi_N = f_N(\vec{\tau}_N)$ be bounded random variables. Calculate using (4.18) and (4.19):

$$\begin{aligned}
E\xi_1\xi_2\cdots\xi_N &= E\xi_2\cdots\xi_N E[\xi_1 \,|\, \hat{\mathcal{F}}^1] = E\xi_2\cdots\xi_N E'[\xi_1 \frac{\rho_1}{\hat{\rho}_1} \,|\, \hat{\mathcal{F}}^1] \\
&= E\xi_3\cdots\xi_N E'[\xi_2 \frac{\rho_2}{\hat{\rho}_2} E'[\xi_1 \frac{\rho_1}{\hat{\rho}_1} \,|\, \hat{\mathcal{F}}^1] \,|\, \hat{\mathcal{F}}^2] \\
&= E\xi_3\cdots\xi_N E'[\xi_1\xi_2 \frac{\rho_1}{\hat{\rho}_1}\frac{\rho_2}{\hat{\rho}_2} \,|\, \hat{\mathcal{F}}^2] \\
&= EE'[\xi_1\cdots\xi_N \prod_{j=1}^{N} \frac{\rho_i}{\hat{\rho}_i} \,|\, \hat{\mathcal{F}}^N] \\
&= EE'[\xi_1\cdots\xi_N \prod_{j=1}^{n} \frac{\rho_i}{\hat{\rho}_i} \,|\, \hat{\mathcal{F}}^n],
\end{aligned}$$

for any $n > N$.

By Fatou lemma, Lemma 4.8, and the fact that $\hat{\mathcal{F}}^\infty$ is trivial we get

$$E(\xi_1\xi_2\cdots\xi_N) \geq EE'[\xi_1\cdots\xi_N\left(\prod_{j=1}^{\infty}\frac{\rho_i}{\hat{\rho}_i}\right)\wedge M\,|\,\hat{\mathcal{F}}^\infty] = E'(\xi_1\cdots\xi_N\rho\wedge M),$$

for all $M > 0$. It follows that

$$E'\rho \leq 1 \quad \text{and} \quad E(\xi_1\xi_2\cdots\xi_N) \geq E'(\xi_1\cdots\xi_N\rho). \tag{4.20}$$

The very same calculations with P and P' exchanged yield

$$E\frac{1}{\rho} \leq 1 \quad \text{and} \quad E'\xi_1\xi_2\cdots\xi_N \geq E\xi_1\cdots\xi_N\frac{1}{\rho}.$$

Thus $P \sim P'$ and $P = \rho\cdot P'$. The proof of the first part of the theorem is complete.

Assume $P = \rho\cdot P'$. We will first prove

$$E'g(\vec{\tau}_k,\vec{\tau}_{k+1},\dots) = E'\left(\prod_{j=1}^{k-1}\frac{\hat{\rho}_j}{\rho_j}\right)g(\vec{\tau}_k,\vec{\tau}_{k+1},\dots), \tag{4.21}$$

for any k and measurable bounded function g.

Notice that ρ_k and $\hat{\rho}_k$ are $\hat{\mathcal{F}}^{k-1}$-adapted and calculate

$$\begin{aligned}
E'g(\vec{\tau}_k,\vec{\tau}_{k+1},\dots) &= Eg(\vec{\tau}_k,\vec{\tau}_{k+1},\dots)E[\frac{\hat{\rho}_1}{\rho_1}\,|\,\hat{\mathcal{F}}^1]\prod_{i=2}^{\infty}\frac{\hat{\rho}_i}{\rho_i} \\
&= Eg(\vec{\tau}_k,\vec{\tau}_{k+1},\dots)\left(\prod_{i=2}^{\infty}\frac{\hat{\rho}_i}{\rho_i}\right)\int\hat{\rho}_1(\vec{\tau}_1,\vec{\tau}_2)P^k(d\vec{\tau}_1) \\
&= Eg(\vec{\tau}_k,\vec{\tau}_{k+1},\dots)\prod_{i=k}^{\infty}\frac{\hat{\rho}_i}{\rho_i}.
\end{aligned}$$

We proved

$$E'g(\vec{\tau}_k, \vec{\tau}_{k+1}, \dots) = E'\left(\prod_{i=1}^{k-1} \frac{\rho_i}{\hat{\rho}_i}\right) g(\vec{\tau}_k, \vec{\tau}_{k+1}, \dots). \tag{4.22}$$

The equation (4.19) is valid if and only if

$$E'f(\vec{\tau}_k, \vec{\tau}_{k+1}, \dots)1_{S_{k+1}} = E'1_{S_{k+1}} \int f(\vec{\tau}_k, \vec{\tau}_{k+1}, \dots)\hat{\rho}_k(\vec{\tau}_k, \vec{\tau}_{k+1})P^k(d\vec{\tau}_k), \tag{4.23}$$

for every $S_{k+1} \in \hat{\mathcal{F}}^k$. However the same calculations as above yield:

$$\begin{aligned} &E'f(\vec{\tau}_k, \vec{\tau}_{k+1}, \dots)1_{S_{k+1}} \\ &\qquad = E'\left(\prod_{i=1}^{k} \frac{\rho_i}{\hat{\rho}_i}\right) 1_{S_{k+1}} \int f(\vec{\tau}_k, \vec{\tau}_{k+1}, \dots)\hat{\rho}_k(\vec{\tau}_k, \vec{\tau}_{k+1})P^k(d\vec{\tau}_k) \\ &\qquad\qquad = E'1_{S_{k+1}} \int f(\vec{\tau}_k, \vec{\tau}_{k+1}, \dots)\hat{\rho}_k(\vec{\tau}_k, \vec{\tau}_{k+1})P^k(d\vec{\tau}_k). \end{aligned}$$

The last equality follows from the formula (4.21), and the fact that the random variable $1_{S_{k+1}} \int f(\vec{\tau}_k, \vec{\tau}_{k+1}, \dots)\hat{\rho}_k(\vec{\tau}_k, \vec{\tau}_{k+1})P^k(d\vec{\tau}_k)$ is $\hat{\mathcal{F}}^k$-adapted. $\square$

The assumption that the density $\rho^{l,k}$ is bounded away from 0 and ∞ is technical and could be somewhat relaxed by assuming that certain entities are integrable.

We have proved that the sequence $(\vec{\tau}_1, \vec{\tau}_2, \dots)$ is a discrete time Markov process. Particularly:

$$\vec{\tau}_1, \dots, \vec{\tau}_{k-1} \amalg_{\vec{\tau}_k} \vec{\tau}_{k+1}, \vec{\tau}_{k+2}, \dots$$

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