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# INTEGRATED BROWNIAN MOTIONS AND EXACT $L_2$-SMALL BALLS

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We will introduce a class of $m$-times integrated Brownian motions. The exact asymptotic expansions for the $L_2$-small ball probabilities will be discussed for members of this class, of which the usual $m$-times integrated Brownian motion is an example. Another example will be what we call an Euler-integrated Brownian motion. We will also find very sharp estimates for the asymptotics of the eigenvalues of the covariance operator of integrated Brownian motions and will, therefore, obtain exact, not just logarithmic, asymptotics.

**1. Introduction.** Let $X_m(t)$, $0 \le t \le 1$, be a usual $m$-times integrated Brownian motion

$$X_m(t) = \int_0^t \int_0^{s_m} \cdots \int_0^{s_2} B(s_1)\, ds_1\, ds_2 \cdots ds_m \tag{1}$$

where $B(s)$ is a standard Brownian motion and $m \ge 0$ is an integer. In this paper we will be interested in understanding the behavior of the probability that $X_m(t)$ stays in a small ball of radius $\varepsilon$ around the origin

$$P\left(\|X_m\| \le \varepsilon\right). \tag{2}$$

As $\varepsilon$ tends to 0, this probability clearly tends to 0—the question is, *at what rate*?

Questions of this type fall into the realm of what has come to be called "small ball" probabilities. The asymptotic analysis of small ball probabilities for Gaussian processes has received much attention in the literature [see, e.g., the recent survey of Li and Shao (2001)].

To be specific, we will be interested in understanding the asymptotic behavior of (2) with the $L_2$ norm. This question was previously studied by Khoshnevisan and Shi (1998) and Chen and Li (2003). They prove the exact logarithmic asymptotics of the small ball probability of $m$-times integrated Brownian motion for $m = 1$ and $m \ge 1$ respectively. Actually Khoshnevisan and Shi (1998) achieve more. They find an explicit representation for the Laplace transform

$$E \exp\left(-\frac{\theta^2}{2} \int_0^1 X_1^2(t)\, dt\right) = \left(\frac{2}{\cosh^2 \sqrt{\theta/2} + \cos^2 \sqrt{\theta/2}}\right)^{1/2}.$$

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Motivated by this result, Chen and Li (2003) obtained the following: as $\varepsilon \to 0^+$,

$$\log P\left(\int_0^1 X_m^2(t)\,dt \le \varepsilon^2\right) \sim -\frac{2m+1}{2}\left((2m+2)\sin\frac{\pi}{2m+2}\right)^{-(2m+2)/(2m+1)} \varepsilon^{-2/(2m+1)}.$$

That is, they are able to find the exact logarithmic small ball asymptotics for *all* integrated Brownian motions. The calculations performed show that the Laplace transform of $\|X_m\|_2^2$ for $m \ge 2$ is extremely complicated, whereas in the case $m = 0$ and 1 it has a relatively nice form.

In this paper we extend their result beyond the logarithmic asymptotics, and, more importantly, we show that our result is true for a larger class of processes we call general $m$-times integrated Brownian motions. Informally, a general $m$-times integrated Brownian motion is a $\mathcal{C}[0,1]$ process defined as in (1) with $B_s$ replaced by $B_{1-s}$ or with some of the limits of integration changed from $(0, s_k)$ to $(s_k, 1)$. Formally, we define the general $m$-times integrated Brownian motion as follows. Consider the following Sturm–Liouville problem:

$$\lambda f^{(2m+2)}(t) = (-1)^{m+1} f(t), \qquad 0 < t < 1, \tag{3}$$

with the boundary conditions

$$f(t_0) = f'(t_1) = f''(t_2) = \cdots = f^{(2m+1)}(t_{2m+1}) = 0, \tag{4}$$

where

$$t_j \in \{0, 1\} \qquad \text{for each } j, \quad \text{and} \quad \sum_{j=0}^{2m+1} t_j = m+1. \tag{5}$$

We say that the boundary conditions of (4) are *antisymmetric* if for each $0 \le j \le m$, $t_j = 1 - t_{2m+1-j}$. Integrating (3) with antisymmetric boundary conditions leads to the following integral equation.

$$\lambda f(t) = Af(t) = \int_0^1 K(s,t) f(s)\,ds, \tag{6}$$

where $K(s,t)$ is a (uniquely determined) symmetric and positive definite kernel. Call the unique centered Gaussian process $X(t)$ on $[0,1]$ with the covariance function $K(s,t)$ a *general $m$-times integrated Brownian motion*. We will call $A$ the covariance operator of $X(t)$. It is easy to see that if the boundary conditions are not antisymmetric, then (3) and (4) do not lead to a symmetric, positive definite kernel and will therefore lack a probabilistic interpretation.

There are two important special cases. When the boundary conditions (4) are chosen so that $t_j = 0$ for $0 \le j \le m$ and $t_j = 1$ for $j > m$, then the resulting covariance kernel is

$$K(s,t) = \frac{1}{(m!)^2}\int_0^{s\wedge t} (s-u)^m (t-u)^m\,du.$$

This is just the covariance kernel of the usual $m$-times integrated Brownian motion (1).

Suppose we choose boundary conditions so that $t_j = 0$ for $j$ even and $t_j = 1$ for $j$ odd. The resulting integral equation leads one to the kernel

$$\tilde{K}_m(s,t) = \int_0^1 \cdots \int_0^1 (s \wedge s_1)(s_1 \wedge s_2) \cdots (s_m \wedge t)\, ds_1\, ds_2 \cdots ds_m.$$

This kernel has the form [see Chang and Ha (2001), Theorem 2]

$$\tilde{K}_m(s,t) = (-1)^{m+1} \frac{2^{2m}}{(2m+1)!} \left( E_{2m+1}\left(\frac{|s-t|}{2}\right) - E_{2m+1}\left(\frac{s+t}{2}\right)\right)$$

where $E_n(x)$ is the $n$th degree Euler polynomial (see Theorem 1 below). In this case the eigenvalue problem (6) is completely solvable and we get

$$\lambda_n = \left(\frac{1}{\pi(n-1/2)}\right)^{2m+2},$$

which allows us to calculate the exact small ball asymptotics (including all the constants) for this process. We will call the corresponding process the *$m$-times Euler integrated Brownian motion*.

In general, we could have chosen any antisymmetric boundary conditions. It turns out that these processes are not very different from each other from an eigenvalue point of view. We prove the following result:

THEOREM 3. *For the general $m$-times integrated Brownian motion*

$$P\left(\int_0^1 X^2(t)\, dt \le \varepsilon^2\right) \sim C\varepsilon^{(1-k_0(2m+2))/(2m+1)} \exp\{-D_m \varepsilon^{-2/(2m+1)}\}$$

*where*

$$D_m = \frac{2m+1}{2}\left((2m+2)\sin\frac{\pi}{2m+2}\right)^{-(2m+2)/(2m+1)},$$

*$C$ is a positive constant and $k_0$ is an integer. Furthermore, $k_0 = 0$ for $m \le 10$.*

The authors conjecture $k_0 = 0$ for all $m \ge 0$. It is not difficult to show $k_0 = 0$ for $m \le 2$; however, to show $k_0 = 0$ for all $m \ge 0$ would require a general statement about the spectral function for higher-order Sturm–Liouville problems. Roughly speaking, one would need to show the asymptotic equivalence of the spectral functions for certain Sturm–Liouville problems with "permuted" evaluation points.

We prove Theorem 3 by approximating the eigenvalues of the covariance operator of general $m$-times integrated Brownian motion. These approximations will be sharp enough to achieve the exact small ball probability rates. Since centered Gaussian processes are determined uniquely by their covariance kernels, what we shall see is that $m$-times integrated Brownian motions can be understood through the analysis of the corresponding Sturm–Liouville problem.

Let $A$ be the covariance operator of a general $m$-times integrated Brownian motion.

THEOREM 2. *The eigenvalues $\lambda_k$ of $A$ are*

$$\lambda_k = \left(\frac{1}{(k_0 + k - \frac{1}{2})\pi}\right)^{2m+2} + O\left(\frac{1}{k^{2m+3}} \exp\left(-k\pi \sin\left(\frac{\pi}{m+1}\right)\right)\right)$$

*where $k_0 = k_0(A)$ is an integer.*

For example, if we consider the usual one-time integrated Brownian motion it turns out that the eigenvalues $\lambda$ of $A$ satisfy

$$\cosh\left(\frac{1}{\lambda^{1/4}}\right)\cos\left(\frac{1}{\lambda^{1/4}}\right) = -1 \tag{7}$$

[see Courant and Hilbert (1937), page 296, or Freedman (1999)], but as $m$ increases so does the complexity of this defining equation. For example, for $m = 2$, we get

$$\begin{aligned} 4 + 4\cos\left(\frac{1}{\lambda^{1/6}}\right) + \cos^2\left(\frac{1}{\lambda^{1/6}}\right) + 8\cos\left(\frac{1}{2\lambda^{1/6}}\right)\cosh\left(\frac{\sqrt{3}}{2\lambda^{1/6}}\right) \\ + \cos\left(\frac{1}{\lambda^{1/6}}\right)\cosh\left(\frac{\sqrt{3}}{\lambda^{1/6}}\right) = 0. \end{aligned}$$

In the case $m = 1$ it is not difficult to see that in order for a small $\lambda$ to be an eigenvalue the cosine needs to be very close to zero

$$\cos\left(\frac{1}{\lambda^{1/4}}\right) = O\left(\exp\left(-\frac{c}{\lambda^{1/4}}\right)\right).$$

As we shall see later $\lambda_k = (\pi(k - 1/2))^{-4} + O(e^{-k\pi})$ as $k \to \infty$. It turns out that this asymptotic is very sharp, and, with the help of a representation theorem of Sytaya (1974) and comparison theorems of Li (1992), we can get the exact asymptotic behavior of the $L_2$-small ball modulo a constant. In fact, what we will see later is that for $m$ large the complexity of the processes is coming from only the first few eigenvalues after which the eigenvalues behave very much like a fixed function of the zeros of cosine.

The remainder of the paper is organized as follows: In Section 2 we relate the two definitions of the general $m$-times integrated Brownian motion introduced earlier and prove the exact small ball asymptotics of the $m$-times Euler-integrated Brownian motion. In Section 3 we obtain the exact asymptotics of the eigenvalues of the covariance operator $A$ of any general $m$-times integrated Brownian motion, and in Section 4 we state the exact asymptotic formulations for *all* $m$-times integrated Brownian motions and show that $k_0(A) = 0$ for $m \leq 10$. We also provide some compelling graphs which suggest $k_0 = 0$ for all $m$.

**2. General and Euler-integrated Brownian motions.** In this section we first study the connection between the "intuitive" and formal definitions of the general integrated Brownian motion. Then we continue with a more detailed study of the Euler-integrated Brownian motion.

The following notations will be convenient: Let $T_0$ and $T_1$ be operators that act on functions in $L_2[0, 1]$ by

$$T_0 f(t) = \int_0^t f(s)\,ds \quad \text{and} \quad T_1 f(t) = \int_t^1 f(s)\,ds.$$

By Fubini's theorem $\langle f, T_0 g\rangle = \langle T_1 f, g\rangle$, that is, $T_0$ is the adjoint operator to $T_1$.

Let $X(t)$ be a continuous centered Gaussian process with the covariance operator $A$. Then

$$\int_0^t X(s)\,ds = T_0 X(t) \quad \text{and} \quad \int_t^1 X(s)\,ds = T_1 X(t) \tag{8}$$

are continuous centered Gaussian processes. It is easy to check that the covariance operator of $T_i X(t)$ is $T_i A T_{1-i}$.

Let $B_0$ be a standard Brownian motion and $B_1$ the process $B_0(1-s)$. It is well known that the covariance kernel of $B_0$ is $\min(s,t)$. Thus, the covariance operator of $B_i$ is $T_i T_{1-i}$.

Fix an integer $m \geq 0$ and take a sequence $I = \{i_m, i_{m-1}, \dots, i_0\}$, where $i_k \in \{0, 1\}$. In accordance with (8) define $X^I(t) = T_{i_m} \cdots T_{i_1} B_{i_0}(t)$. It follows from the observations made in the previous two paragraphs that the covariance operator of $X^I(t)$ is

$$Af(t) = T_{i_m} \cdots T_{i_0} T_{1-i_0} \cdots T_{1-i_m} f(t). \tag{9}$$

Consider the eigenvalue problem $\lambda f(t) = Af(t)$. Upon successive differentiation we arrive to the following differential equation.

$$\lambda f^{(2m+2)}(t) = (-1)^{m+1} f(t)$$

with boundary conditions

$$\begin{aligned} f(i_m) &= f'(i_{m-1}) = \cdots = f^{(m)}(i_0) \\ &= f^{(m+1)}(1-i_0) = \cdots = f^{(2m+1)}(1-i_m) = 0. \end{aligned}$$

This is exactly the Sturm–Loiuville problem with antisymmetric boundary conditions. The process $X^I(t)$ is then a general $m$-times integrated Brownian motion. This explains the intuition behind the definition, since each $T_i$ is an integral operator. Notice in particular that $X^{\{0,\dots,0\}}$ is the usual $m$-times integrated Brownian motion and $X^{\{0,1,0,1,\dots\}}$ is the Euler-integrated Brownian motion.

In the rest of this section we restrict our attention to the Euler-integrated Brownian motion. We prove the following theorem:

THEOREM 1. *Let $X(t)$ be the $m$-times Euler-integrated Brownian motion. Then*:

(i) *the process $X(t)$ has the following covariance kernel*

$$K_m(s,t)$$

$$= (-1)^m \sum_{j=1}^{m+1} \frac{2^{2(m+1-j)}}{(2j-1)![2(m+1-j)]!}(s\wedge t)^{2j-1} E_{2(m+1-j)}\left(\frac{s\vee t}{2}\right) \tag{10}$$

$$= (-1)^{m+1} \frac{2^{2m}}{(2m+1)!}\left(E_{2m+1}\left(\frac{|t-s|}{2}\right) - E_{2m+1}\left(\frac{t+s}{2}\right)\right), \tag{11}$$

*where $E_n(x)$ is the $n$th Euler polynomial*;

(ii) *the eigenvalues of the covariance operator are $\lambda_k = ((k-\frac{1}{2})\pi)^{-2m-2}$*;

(iii)

$$P\left(\int_0^1 X^2(t)\,dt \le \varepsilon^2\right) \sim C_m \varepsilon^{1/(2m+1)} \exp\{-D_m \varepsilon^{-2/(2m+1)}\}$$

*where*

$$C_m = 2^{(m+1)/2}\left(\frac{2m+2}{(2m+1)\pi}\right)^{1/2}\left[(2m+2)\sin\frac{\pi}{2m+2}\right]^{m+1/(2m+1)}$$

*and*

$$D_m = \frac{2m+1}{2}\left((2m+2)\sin\frac{\pi}{2m+2}\right)^{-(2m+2)/(2m+1)}.$$

We need the following lemma.

LEMMA 1. *Suppose $k \ge 1$ is an integer, then*

$$\prod_{n=1}^{\infty}\left[1+\left(\frac{x}{2n-1}\right)^{2k}\right] = \prod_{j=0}^{k-1}\left[\frac{\cosh(\pi x\sin(\frac{2j+1}{2k}\pi)) + \cos(\pi x\cos(\frac{2j+1}{2k}\pi))}{2}\right]^{1/2}.$$

PROOF. Let $c_j = \exp(\frac{2j+1}{2k}\pi i)$ be the $2k$th roots of $-1$. Then $c_j = -c_{k+j}$, and we have

$$1+\left(\frac{x}{2n-1}\right)^{2k} = \prod_{j=0}^{2k-1}\left[1-\frac{c_j x}{2n-1}\right] = \prod_{j=0}^{k-1}\left[1-\left(\frac{c_j x}{2n-1}\right)^2\right].$$

Using the infinite product expansion of the cosine function

$$\cos(\pi z/2) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{(2n-1)^2}\right),$$

we obtain

$$\begin{aligned}
&\prod_{n=1}^{\infty}\left[1 + \left(\frac{x}{2n-1}\right)^{2k}\right] \\
&\quad = \prod_{j=0}^{k-1}\prod_{n=1}^{\infty}\left[1 - \left(\frac{c_j x}{2n-1}\right)^2\right] = \prod_{j=0}^{k-1}\cos(\pi x c_j/2) \\
&\quad = \prod_{j=0}^{k-1}\left[\cos(\pi x c_j/2)\cos(\pi x c_{k-1-j}/2)\right]^{1/2} \\
&\quad = \prod_{j=0}^{k-1}\left[\frac{\cosh(\pi x \sin(\frac{2j+1}{2k}\pi)) + \cos(\pi x \cos(\frac{2j+1}{2k}\pi))}{2}\right]^{1/2}. \qquad \square
\end{aligned}$$

REMARK 1. When $k = m + 1$ and $x = 2(2\theta)^{1/(2m+2)}/\pi$ we obtain the exact Laplace transform $E\exp(-\theta\int_0^1 X^2(s)\,ds)$ of the squared $L_2$ norm of the $m$-times Euler-integrated Brownian motion.

PROOF OF THEOREM 1. (i) The authors proved the formulation (10) before we discovered the formulation (11) in Chang and Ha (2001). We just need to check that either kernel representation for $A$ will lead to the correct differential equation. The following facts regarding Euler polynomials will be useful in this regard [see Abramowitz and Stegun (1972), page 805]: For $n \geq 0$ an integer,

$$E_n'(x) = nE_{n-1}(x) \quad \text{and} \quad E_{2n}(0) = 0.$$

The verification is straightforward and we omit the calculation.

(ii) Notice that (9) implies that the covariance operator

$$Af(t) = C^{m+1}f(t)$$

where $C$ is the covariance operator of the standard Brownian motion. Recall that the operator $C$ defined on $L_2[0,1]$ has eigenvalues $\nu_k = (\pi(k - 1/2))^{-2}$ for $k \geq 1$ and corresponding eigenfunctions $\phi_k(t) = \sqrt{2}\sin(\pi(k - 1/2)t)$. The eigenvalues and eigenfunctions of $A$ now follow via the relation

$$A\phi(t) = C^{m+1}\phi(t) = \nu^{m+1}\phi(t).$$

That is, the eigenvalues of $A$ are $\lambda_k = \nu_k^{m+1} = (\pi(k - 1/2))^{-(2m+2)}$ and we have exactly the same eigenfunctions.

(iii) It follows from the Karhunen–Loève expansion of $X(t)$ that

$$P\left(\int_0^1 X^2(t)\,dt \le \varepsilon^2\right) = P\left(\sum_{k=1}^{\infty} \lambda_k \xi_k^2 \le \varepsilon^2\right)$$

where $\xi_k$ are independent $N(0,1)$ random variables. We use the following result from Sytaya (1974):

$$P\left(\sum_{k=1}^{\infty} \lambda_k \xi_k^2 \le \varepsilon^2\right)$$
$$\sim \left(4\pi \sum_{k=1}^{\infty} \left(\frac{\lambda_k \gamma}{1+2\lambda_k\gamma}\right)^2\right)^{-1/2} \exp\left\{\varepsilon^2\gamma - \frac{1}{2}\sum_{k=1}^{\infty} \log(1+2\lambda_k\gamma)\right\},$$

where $\gamma$ satisfies the following relation:

$$\varepsilon^2 = \sum_{k=1}^{\infty} \frac{\lambda_k}{1+2\lambda_k\gamma}.$$

Define

$$h(\gamma) = \tfrac{1}{2}\log \prod_{k=1}^{\infty} (1+2\lambda_k\gamma).$$

Then

$$\varepsilon^2\gamma - \tfrac{1}{2}\sum_{k=1}^{\infty}\log(1+2\lambda_k\gamma) = \gamma h'(\gamma) - h(\gamma),$$
$$4\pi \sum_{k=1}^{\infty}\left(\frac{\lambda_k\gamma}{1+2\lambda_k\gamma}\right)^2 = -2\pi\gamma^2 h''(\gamma).$$

Applying Lemma 1 with $x = 2(2\gamma)^{1/(2m+2)}/\pi$, we have

$$\begin{aligned} h(\gamma) &= \frac{1}{2}\log \prod_{k=1}^{\infty}\left(1 + \left(\frac{2(2\gamma)^{1/(2m+2)}/\pi}{2k-1}\right)^{2m+2}\right) \\ &= \frac{1}{4}\log \prod_{j=0}^{m}\left[\cosh\left(2(2\gamma)^{1/(2m+2)}\sin\left(\frac{2j+1}{2m+2}\pi\right)\right)\right. \\ &\qquad\left. + \cos\left(2(2\gamma)^{1/(2m+2)}\cos\left(\frac{2j+1}{2m+2}\pi\right)\right)\right] - \frac{m+1}{4}\log 2. \end{aligned}$$

It is not hard to check that

$$h(\gamma) = 2^{-1}(2\gamma)^{1/(2m+2)} \csc\left(\frac{\pi}{2m+2}\right) - \frac{m+1}{2}\log 2 + O\left(\exp\left(-2(2\gamma)^{1/(2m+2)} \sin\left(\frac{\pi}{2m+2}\right)\right)\right),$$

$$\gamma h'(\gamma) = \frac{1}{2m+2} 2^{-1}(2\gamma)^{1/(2m+2)} \csc\left(\frac{\pi}{2m+2}\right) + O\left(\exp\left(-2(2\gamma)^{1/(2m+2)} \sin\left(\frac{\pi}{2m+2}\right)\right)\right)$$

and

$$\gamma^2 h''(\gamma) = -\frac{2m+1}{2m+2}\frac{1}{2m+2} 2^{-1}(2\gamma)^{1/(2m+2)} \csc\left(\frac{\pi}{2m+2}\right) + O\left(\exp\left(-2(2\gamma)^{1/(2m+2)} \sin\left(\frac{\pi}{2m+2}\right)\right)\right).$$

On the other hand, by the definition of $\gamma$, $\varepsilon^2 = h'(\gamma)$. This implies that

$$(2\gamma)^{1/(2m+2)} = \left[(2m+2)\sin\left(\frac{\pi}{2m+2}\right)\right]^{-1/(2m+1)} \varepsilon^{-2/(2m+1)} + O\left(\exp(-C\varepsilon^{-2/(2m+1)})\right), \tag{12}$$

for some positive constant $C$. Thus,

$$\exp\left(\gamma h'(\gamma) - h(\gamma)\right) \sim 2^{(m+1)/2} \exp\left(D_m \varepsilon^{-2/(2m+1)}\right)$$

and

$$[-2\pi\gamma^2 h''(\gamma)]^{-1/2} \sim \left(\frac{2m+2}{(2m+1)\pi}\right)^{1/2} \left[(2m+2)\sin\left(\frac{\pi}{2m+2}\right)\right]^{(m+1)/(2m+1)} \varepsilon^{1/(2m+1)}.$$

The statement of the theorem now follows. □

**3. Eigenvalue approximations.** In this section we obtain very sharp asymptotics of the eigenvalues for the covariance operator $A$ of all general $m$-times integrated Brownian motions.

THEOREM 2. *The eigenvalues $\lambda_k$ of $A$ are*

$$\lambda_k = \left(\frac{1}{(k_0 + k - \frac{1}{2})\pi}\right)^{2m+2} + O\left(\frac{1}{k^{2m+3}} \exp\left(-k\pi \sin\left(\frac{\pi}{m+1}\right)\right)\right)$$

*where $k_0 = k_0(A)$ is an integer.*

Notice that the error term is of exponential order. The only setback is the presence of the integer $k_0$, which we are not able to evaluate for general $m$. However, we shall see that $k_0 = 0$ for all $m \leq 10$ (see Section 4).

It is worth mentioning that, in the case of Brownian motion (i.e., $m = 0$), our formula recovers the well-known exact solution. In the case of $m = 1$ we sharpen the result of Freedman (1999), who proved that $\lambda_k \sim (k\pi)^{-4}$. In fact, in his paper Freedman defines a general class of priors which is an example that shows for a statistical problem with infinitely many parameters the Bayesian and frequentist methods lead to totally different results. Our result shows that general $m$-times integrated Brownian motions fall exactly into the framework of his example.

PROOF. For simplicity we first consider the usual $m$-times integrated Brownian motion. We need to analyze the following Sturm–Liouville equation:

$$\lambda f^{(2m+2)}(t) = (-1)^{m+1} f(t) = (i)^{2m+2} f(t)$$

with boundary conditions

$$f^{(k)}(0) = f^{(m+1+k)}(1) = 0$$

for $k = 0, 1, \ldots, m$. The eigenfunctions are the nontrivial functions of the form

$$f(t) = \sum_{j=0}^{2m+1} c_j e^{\alpha_j t}$$

with $\alpha_j = \lambda^{-1/(2m+2)} i\omega_j$ and $\omega_j = \exp(\frac{j\pi}{m+1} i)$ satisfying the boundary conditions. The idea is to find the values of $\lambda$ which make nontrivial solutions possible. Taking derivatives, plugging in the boundary conditions, and dividing the $k$th row by $(\lambda^{-1/(2m+2)} i)^k$ leads to the following $(2m+2) \times (2m+2)$ matrix:

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_{2m+1} \\ \vdots & \vdots & \cdots & \vdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_{2m+1}^m \\ \omega_0^{m+1} e^{\alpha_0} & \omega_1^{m+1} e^{\alpha_1} & \cdots & \omega_{2m+1}^{m+1} e^{\alpha_{2m+1}} \\ \vdots & \vdots & \cdots & \vdots \\ \omega_0^{2m+1} e^{\alpha_0} & \omega_1^{2m+1} e^{\alpha_1} & \cdots & \omega_{2m+1}^{2m+1} e^{\alpha_{2m+1}} \end{pmatrix}.$$

The key observation is that $\lambda$ is an eigenvalue if and only if $\det M = 0$.

We claim $\det M$ is either real or pure imaginary. To see this, we notice $\overline{\omega_j} = -\omega_{m+1-j}$ and $\overline{e^{\alpha_j}} = e^{\alpha_{m+1-j}}$. Furthermore, $\overline{M} = DMR$ where $D$ is a $(2m+2) \times (2m+2)$ matrix with entries $d_{kj} = (-1)^k \delta_{kj}$; $R$ is a $(2m+2) \times (2m+2)$ matrix with entries $r_{kj} = 1$ if $k + j = m + 3$ or $k + j = 3m + 5$, and $r_{kj} = 0$ otherwise. This implies that $\det \overline{M} = (-1)^m \det M$. Thus, $\det M = 0$ if and only if $\operatorname{Re}(i^m \det M) = 0$.

Multiply the last $m$ columns of $M$ by $e^{\alpha_1}$, $e^{\alpha_2}$, $\dots$, $e^{\alpha_m}$ respectively, and use $\alpha_j = -\alpha_{m+1+j}$ to obtain the following matrix $N$:

$$N = \begin{pmatrix} 1 & \cdots & 1 & 1 & e^{\alpha_1} & \cdots & e^{\alpha_m} \\ \omega_0 & \cdots & \omega_m & \omega_{m+1} & \omega_{m+2}e^{\alpha_1} & \cdots & \omega_{2m+1}e^{\alpha_m} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \omega_0^m & \cdots & \omega_m^m & \omega_{m+1}^m & \omega_{m+2}^m e^{\alpha_1} & \cdots & \omega_{2m+1}^m e^{\alpha_m} \\ \omega_0^{m+1}e^{\alpha_0} & \cdots & \omega_m^{m+1}e^{\alpha_m} & \omega_{m+1}^{m+1}e^{-\alpha_0} & \omega_{m+2}^{m+1} & \cdots & \omega_{2m+1}^{m+1} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \omega_0^{2m+1}e^{\alpha_0} & \cdots & \omega_m^{2m+1}e^{\alpha_m} & \omega_{m+1}^{2m+1}e^{-\alpha_0} & \omega_{m+2}^{2m+1} & \cdots & \omega_{2m+1}^{2m+1} \end{pmatrix}.$$

Note that $e^{\alpha_1} \cdot e^{\alpha_2} \cdots e^{\alpha_m} = \exp(-\csc(\frac{\pi}{2m+2}) \sin(\frac{m\pi}{2m+2})\lambda^{-1/(2m+2)})$. Thus, $\det M = 0$ if and only if $\mathrm{Re}(i^m \det N) = 0$. Further notice that for $1 \le j \le m$,

$$|e^{\alpha_j}| = \exp\biggl(-\lambda^{-1/(2m+2)} \sin\biggl(\frac{j\pi}{m+1}\biggr)\biggr) \to 0 \qquad \text{as } \lambda \to 0^+.$$

Therefore, we have

$$\det N = \det N_0 + O\biggl(\exp\biggl(-\lambda^{-1/(2m+2)} \sin\biggl(\frac{\pi}{m+1}\biggr)\biggr)\biggr), \tag{13}$$

where $N_0$ is the matrix obtained from $N$ by replacing all the entries containing $e^{\alpha_j}$, $1 \le j \le m$, with 0. That is,

$$N_0 = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ \omega_0 & \omega_1 & \cdots & \omega_m & \omega_{m+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_m^m & \omega_{m+1}^m & 0 & \cdots & 0 \\ \omega_0^{m+1}e^{\alpha_0} & 0 & \cdots & 0 & \omega_{m+1}^{m+1}e^{-\alpha_0} & \omega_{m+2}^{m+1} & \cdots & \omega_{2m+1}^{m+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \omega_0^{2m+1}e^{\alpha_0} & 0 & \cdots & 0 & \omega_{m+1}^{2m+1}e^{-\alpha_0} & \omega_{m+2}^{2m+1} & \cdots & \omega_{2m+1}^{2m+1} \end{pmatrix}.$$

It is now easy to see that $\det N_0 = e^{-\alpha_0}\det(U) \cdot \det(V) - e^{\alpha_0}\det(U') \cdot \det(V')$, where

$$U = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_m \\ \vdots & \vdots & \cdots & \vdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_m^m \end{pmatrix}, \qquad V = \begin{pmatrix} \omega_{m+1}^{m+1} & \omega_{m+2}^{m+1} & \cdots & \omega_{2m+1}^{m+1} \\ \omega_{m+1}^{m+2} & \omega_{m+2}^{m+2} & \cdots & \omega_{2m+1}^{m+2} \\ \vdots & \vdots & \cdots & \vdots \\ \omega_{m+1}^{2m+1} & \omega_{m+2}^{2m+1} & \cdots & \omega_{2m+1}^{2m+1} \end{pmatrix},$$

$U'$ is the matrix obtained by replacing the first column of $U$ by the column vector $(1\ \omega_{m+1}\ \dots\ \omega_{m+1}^m)^T$, and $V'$ is the matrix obtained by replacing the first column of $V$ by the column vector $(\omega_0^{m+1}\ \omega_0^{m+2}\ \dots\ \omega_0^{2m+1})^T$.

We will now manipulate $\det(U')$ [resp. $\det(V')$] in such a way that we get a constant times $\det(U)$ [resp. $\det(V)$]. Notice that the first column of the matrices $U'$ and $V'$ is a natural successor of the last column. Thus, by factoring $\omega_1^j = \omega_j$ from the $j$th row of $U'$ and factoring $\omega_1^{m+1+j} = \omega_{m+1+j}$ from the $j$th row of $V'$ for each $0 \le j \le m$, we obtain

$$\begin{aligned}\det N_0 &= e^{-\alpha_0}\det(U)\cdot\det(V) - e^{\alpha_0}\det(U)\cdot\det(V)\prod_{j=0}^{2m+1}\omega_j \\ &= \det(U)\det(V)(e^{-\alpha_0}+e^{\alpha_0}).\end{aligned} \tag{14}$$

Further, $\det(U)\cdot\det(V)\neq 0$. By checking the conjugate, we have $\det\overline{N_0} = (-1)^m\det N_0$. So, $i^m\det(U)\cdot\det(V)$ is a nonzero real number. Thus, by (13),

$$\cos(\lambda^{-1/(2m+2)}) = O\left(\exp\left(-\lambda^{-1/(2m+2)}\sin\left(\frac{\pi}{m+1}\right)\right)\right), \tag{15}$$

and the right-hand side of (15) is real.

Choose $\lambda_0$ small enough so that absolute value of the right-hand side of (15) is less than 1. If $\lambda_n$ is an eigenvalue, applying the intermediate value theorem for real continuous functions, we obtain, after some simple algebra,

$$\lambda_n = \left[(k-1/2)\pi\right]^{-2m-2} + O\left(\frac{1}{k^{2m+3}}\exp\left(-k\pi\sin\left(\frac{\pi}{m+1}\right)\right)\right)$$

for some integer $k$.

Since the intermediate value theorem gives only the existence and not the uniqueness of the root, we have to ensure that the $\det N$ as a function of $\lambda$ is monotone in an appropriate neighborhood of $((k-1/2)\pi)^{-2m-2}$. To ensure this as well as the fact that these eigenvalues have multiplicity 1, we need to compute $\frac{d}{d\lambda}\det N$. Using similar arguments as in the derivation of (13) we get

$$\begin{aligned}\frac{d}{d\lambda}\det N = \frac{d}{d\lambda}&\det N_0 \\ &+ O\left(\lambda^{-(2m+3)/(2m+2)}\exp\left(-\lambda^{-1/(2m+2)}\sin\left(\frac{\pi}{m+1}\right)\right)\right).\end{aligned}$$

Furthermore, using (14) we obtain

$$\frac{d}{d\lambda}\det N_0 = \det(U)\det(V)\sin(\lambda^{-1/(2m+2)})\lambda^{-(2m+3)/(2m+2)}.$$

This implies that there is a $0 < \lambda_0' \le \lambda_0$ such that, at each eigenvalue $\lambda < \lambda_0'$ that is exponentially close to $((k-1/2)\pi)^{-2m-2}$, $\frac{d}{d\lambda}\det N$ is exponentially close to $(-1)^k \det(U)\det(V)[(k-1/2)\pi]^{2m+3}$, which is bounded away from 0. The local monotonicity and the fact that the eigenvalues have multiplicity 1 follows.

Since the covariance operator $A$ is positive and compact, $A$ has only finitely many eigenvalues bigger than $\lambda_0'$. We conclude that

$$\lambda_k = \left[(k_0+k-1/2)\pi\right]^{-2m-2} + O\left(\frac{1}{k^{2m+3}}\exp\left(-k\pi\sin\left(\frac{\pi}{m+1}\right)\right)\right)$$

for some integer $k_0$.

For the general $m$-times integrated Brownian motion defined in the Introduction, the corresponding matrix $M$ is changed into $M = (m_{kj})$, where $m_{kj} = \omega_j^k e^{t_k \alpha_j}$, and where $t_k$ satisfy (4) and (5). By checking the conjugate, one can see the determinant of $M$ is still either real or pure imaginary. To estimate the determinant, we apply a similar approximation. After some appropriate row permutations, we obtain

$$N_0 = \begin{pmatrix} \omega_0^{n_0} & \omega_1^{n_0} \cdots \omega_m^{n_0} & \omega_{m+1}^{n_0} & 0 & \cdots & 0 \\ \omega_0^{n_1} & \omega_1^{n_1} \cdots \omega_m^{n_1} & \omega_{m+1}^{n_1} & 0 & \cdots & 0 \\ \vdots & \vdots \ \cdots \ \vdots & \vdots & \vdots & \cdots & \vdots \\ \omega_0^{n_m} & \omega_1^{n_m} \cdots \omega_m^{n_m} & \omega_{m+1}^{n_m} & 0 & \cdots & 0 \\ \omega_0^{l_0} e^{\alpha_0} & 0 \ \cdots \ 0 & \omega_{m+1}^{l_0} e^{-\alpha_0} & \omega_{m+2}^{l_0} & \cdots & \omega_{2m+1}^{l_0} \\ \vdots & \vdots \ \cdots \ \vdots & \vdots & \vdots & \cdots & \vdots \\ \omega_0^{l_m} e^{\alpha_0} & 0 \ \cdots \ 0 & \omega_{m+1}^{l_m} e^{-\alpha_0} & \omega_{m+2}^{l_m} & \cdots & \omega_{2m+1}^{l_m} \end{pmatrix},$$

where $n_j$ and $l_j$ satisfy $\{n_0, n_1, \ldots, n_m\} = \{i : t_i = 0\}$ and $\{l_0, l_1, \ldots, l_m\} = \{i : t_i = 1\}$. (Notice that antisymmetry implies the number of 0's and 1's is the same.) It is not hard to see that $\det N_0 = e^{-\alpha_0}\det(U)\cdot\det(V) - e^{\alpha_0}\det(U')\cdot\det(V')$, where

$$U = \begin{pmatrix} \omega_0^{n_0} & \omega_1^{n_0} & \cdots & \omega_m^{n_0} \\ \omega_0^{n_1} & \omega_1^{n_1} & \cdots & \omega_m^{n_1} \\ \vdots & \vdots & \cdots & \vdots \\ \omega_0^{n_m} & \omega_1^{n_m} & \cdots & \omega_m^{n_m} \end{pmatrix}, \qquad V = \begin{pmatrix} \omega_{m+1}^{l_0} & \omega_{m+2}^{l_0} & \cdots & \omega_{2m+1}^{l_0} \\ \omega_{m+1}^{l_1} & \omega_{m+2}^{l_1} & \cdots & \omega_{2m+1}^{l_1} \\ \vdots & \vdots & \cdots & \vdots \\ \omega_{m+1}^{l_m} & \omega_{m+2}^{l_m} & \cdots & \omega_{2m+1}^{l_m} \end{pmatrix},$$

$U'$ is the matrix obtained by replacing the first column of $U$ by the column vector $(\omega_{m+1}^{n_0}\ \omega_{m+1}^{n_1}\ \ldots\ \omega_{m+1}^{n_m})^T$, and $V'$ is the matrix obtained by replacing the first column of $V$ by the column vector $(\omega_0^{l_0}\ \omega_0^{l_1}\ \ldots\ \omega_0^{l_m})^T$. By factoring $\omega_{n_j}$ from the $j$th row of $U'$ and factoring $\omega_{l_j}$ from the $j$th row of $V'$ for each $0 \le j \le m$, we obtain

$$\begin{aligned}\det N_0 &= e^{-\alpha_0}\det(U)\cdot\det(V) - e^{\alpha_0}\det(U)\cdot\det(V)\prod_{j=0}^{2m+1}\omega_j\\ &= \det(U)\det(V)(e^{-\alpha_0}+e^{\alpha_0}).\end{aligned}$$

Notice $\det(U)\cdot\det(V)\neq 0$. Thus (15) remains unchanged. The rest of the proof is also similar. $\square$

**4. Exact small ball rates.** We now restate the main theorem of this paper.

THEOREM 3. *For the general $m$-times integrated Brownian motion*

$$P\biggl(\int_0^1 X^2(t)\,dt \le \varepsilon^2\biggr) \sim C\varepsilon^{(1-k_0(2m+2))/(2m+1)}\exp\{-D_m\varepsilon^{-2/(2m+1)}\}$$

*where*

$$D_m = \frac{2m+1}{2}\biggl((2m+2)\sin\frac{\pi}{2m+2}\biggr)^{-(2m+2)/(2m+1)},$$

*$C$ is a positive constant, and $k_0$ is an integer. Furthermore, $k_0 = 0$ for $m \le 10$.*

PROOF. Let $a_k$ denote the $k$th eigenvalue of $A$, $b_k = ((k-\frac{1}{2}+k_0)\pi)^{-2m-2}$ and $c_k = ((k-\frac{1}{2})\pi)^{-2m-2}$. In the definition of $b_k$ if $k_0 < 0$ then define $b_1,\dots,b_{-k_0}$ as 1. From Li's comparison theorem [see Li (1992), Theorem 2] we conclude

$$P\biggl(\sum_{k=1}^{\infty} a_k\xi_k^2 \le \varepsilon^2\biggr) \sim C\,P\biggl(\sum_{k=1}^{\infty} b_k\xi_k^2 \le \varepsilon^2\biggr), \tag{16}$$

where $C = \prod_{k=1}^{\infty}(b_k/a_k)^{1/2}$ is a finite positive constant. In order to prove the theorem we need to get from the $b_k$ to the $c_k$ which can be done by Li (1992), Theorem 3. Namely,

$$P\biggl(\sum_{k=1}^{\infty} b_k\xi_k^2 \le \varepsilon^2\biggr) \sim C\gamma^{k_0/2}P\biggl(\sum_{k=1}^{\infty} c_k\xi_k^2 \le \varepsilon^2\biggr) \tag{17}$$

where $C$ is a positive constant and $\gamma \sim C'\varepsilon^{-4(m+1)/(2m+1)}$ as in (12). Hence, by combining Theorem 1(iii) and (16) and (17) we conclude

$$P\biggl(\sum_{k=1}^{\infty} a_k\xi_k^2 \le \varepsilon^2\biggr) \sim C\varepsilon^{(1-k_0(2m+2))/(2m+1)}\exp\{-D_m\varepsilon^{-2/(2m+1)}\}.$$

This proves the first statement.

For each $m$ there are $2^{m+1}$ general integrated Brownian motions. Since each of these processes has a conjugate process, that is, since the covariance operator of the process $X^{\{i_0,\dots,i_m\}}$ has the same eigenvalues as the one for $X^{\{1-i_0,\dots,1-i_m\}}$, it suffices to consider only $2^m$ boundary conditions.

Case $m = 0$: Trivially true.

Case $m = 1$: There are essentially only 2 integrated Brownian motions: the usual, and the Euler-integrated. For the usual integrated Brownian motion, we have, from equation (7), defining $\rho = \lambda^{-1/4}$, the function

$$f_0(\rho) = \cos\rho \cosh\rho + 1.$$

Then it is easy to check that $f_0(\rho) > 0$ and $f_0(\rho) < 0$ for $\rho \in (2k\pi, (2k+1)\pi)$ and $\rho \in ((2k-1)\pi, 2k\pi)$, respectively, for $k = 0, 1, 2, \dots$. Furthermore, $f_0'(\rho) < 0$ and $f_0'(\rho) > 0$ for $\rho \in (2k\pi, (2k+1)\pi)$ and $\rho \in ((2k-1)\pi, 2k\pi)$, respectively. Therefore, $f_0$ has unique 0's in the intervals $(k\pi, (k+1)\pi)$ for $k = 0, 1, 2, \dots$. For the Euler-integrated process there is nothing to show. This proves $m = 1$.

Case $m = 2$: There are essentially 4 processes we need to consider: the usual, the Euler-integrated, and the processes $X^{\{0,1,1\}}$ and $X^{\{0,0,1\}}$. Respectively, the determinant functions are ($\rho = \lambda^{-1/6}$):

$$\begin{aligned}
g_1(\rho) &= 48 + 48\cos(\rho) + 12\cos^2(\rho) \\
&\qquad + 96\cos\left(\frac{\rho}{2}\right)\cosh\left(\frac{\sqrt{3}\rho}{2}\right) + 12\cos(\rho)\cosh(\sqrt{3}\rho), \\
g_2(\rho) &= 108\cos^2(\rho) + 108\cos(\rho)\cosh(\sqrt{3}\rho), \\
g_3(\rho) &= 48\cos(\rho) + 24\cos(2\rho) \\
&\qquad + 96\cos\left(\frac{\rho}{2}\right)\cosh\left(\frac{\sqrt{3}\rho}{2}\right) + 48\cos(\rho)\cosh(\sqrt{3}\rho), \\
g_4(\rho) &= 48\cos(\rho) + 24\cos(2\rho) \\
&\qquad + 96\cos\left(\frac{\rho}{2}\right)\cosh\left(\frac{\sqrt{3}\rho}{2}\right) + 48\cos(\rho)\cosh(\sqrt{3}\rho).
\end{aligned}$$

Remarkably, we find that $g_3 = g_4$. So we have reduced our work to showing $k_0 = 0$ for only $g_1$ and $g_3$, since we get the Euler-integrated process for free. The proof is now a straightforward calculus exercise, and we omit the details.

The authors ran computer calculations that show $k_0 = 0$ for $m \leq 10$, that is, $\lambda_k \approx ((k - \frac{1}{2})\pi)^{-2m-2}$. The main idea is captured in Figure 1. We display $K \det(N)(\rho)$ as a function of $\rho$ ($\rho = \lambda^{-1/(2m+2)}$) for a few usual $m$-times integrated Brownian motions. The purpose of the $K$ is solely to scale the graph to reasonably fit on the page. Each zero of the graph of the determinant corresponds to an eigenvalue of $A$ to the power of $-1/(2m+2)$. In Section 3 we proved that the scaled determinant should behave like a cosine for large $\rho$. We can see this in the picture. However, the most important feature that the picture shows is the following: There exists an integer $l^* > 0$ such that the number of 0's of $K \det(N)$ to the left of $l\pi$ and the number of 0's of cosine to the left of $l\pi$ is the same for all integer $l \geq l^*$.

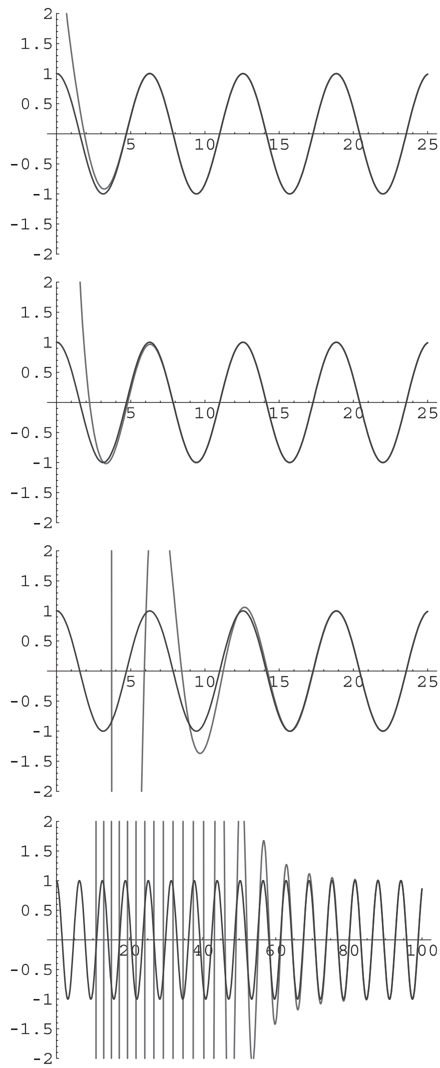


FIG. 1. *Scaled determinant and the limiting cosine of usual integrated Brownian motion with* $m = 1, 2, 6, 25$.

By checking this, for all $m \leq 10$, we can conclude that $k_0 = 0$. The authors conjecture that $k_0 = 0$ for a general $m$. $\square$

One more remark is in order. It seems intuitively clear that, among the general $m$-times integrated Brownian motions, the usual integrated Brownian motions have the largest $L_2$-small ball probabilities while the Euler-integrated Brownian motions have the smallest $L_2$-small ball probabilities. The authors could not formally prove this observation; however, it is supported by computer calculations.

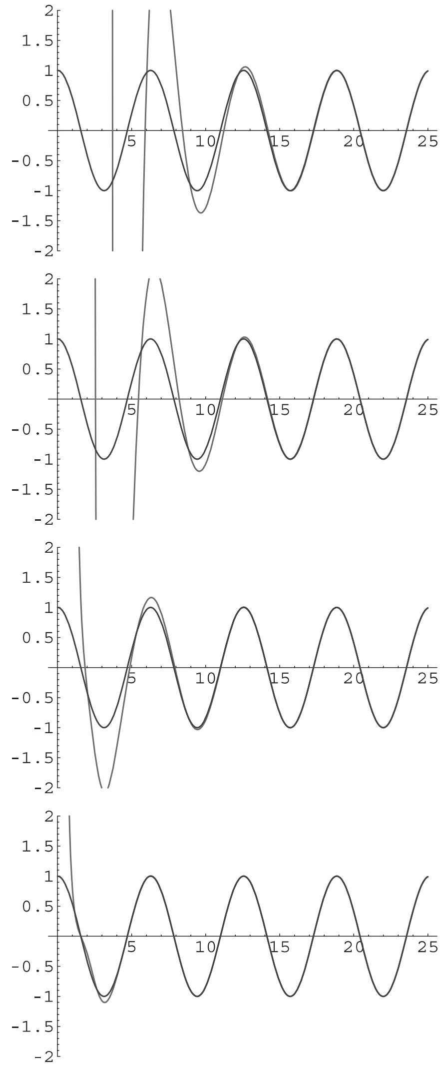


FIG. 2. *Scaled determinant and the limiting cosine for* $X^{\{0,0,0,0,0,0,0\}}$, $X^{\{0,1,0,0,0,0,0\}}$, $X^{\{0,1,0,1,0,0,0\}}$, $X^{\{0,1,0,1,0,1,0\}}$, *respectively.*

For an illustration we have in Figure 2 graphs of $K\det(N)(\rho)$ for several six-times integrated Brownian motions. Notice in particular that the first few 0's of $K\det(N)(\rho)$ are decreasing as we progress from the usual integrated Brownian motion to the Euler integrated Brownian motion. This observation would imply that for any general $m$-times integrated Brownian motion $0 \le k_0 \le k_0^u$, where $k_0^u$ is the $k_0$ associated with the usual integrated Brownian motion. This is in agreement with our belief that in all cases $k_0 = 0$.

NOTE ADDED IN PROOF. Since the acceptance of this article, the authors along with Professor Tzong-Yow Lee [Gao, Hannig, Lee and Torcaso (2003)] have been able to establish that, indeed, for any generalized integrated Brownian motion $k_0 = 0$ for all integers $m \geq 0$.

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