Fiducial Intervals for Variance Components in an Unbalanced Two-Component Normal Mixed Linear Model

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In this article we propose a new method for constructing confidence intervals for σ_α^2, σ_ε^2, and the intraclass correlation $\rho = \sigma_\alpha^2/(\sigma_\alpha^2 + \sigma_\varepsilon^2)$ in a two-component mixed-effects linear model. This method is based on an extension of R. A. Fisher's fiducial argument. We conducted a simulation study to compare the resulting interval estimates with other competing confidence interval procedures from the literature. Our results demonstrate that the proposed fiducial intervals have satisfactory performance in terms of coverage probability, as well as shorter average confidence interval lengths overall. We also prove that these fiducial intervals have asymptotically exact frequentist coverage probability. The computations for the proposed procedures are illustrated using real data examples.

KEY WORDS: Fiducial density; Fiducial generalized confidence interval; Unbalanced one-way random-effects model; Variance component.

1. INTRODUCTION

Random-effects and mixed-effects linear models are useful in applications that require accounting for components of variability arising from multiple sources. For example, in animal breeding studies, mixed linear models with two variance components are often used. One variance component accounts for genetic variability, and the other accounts for variability due to environmental factors. In industrial applications where one is interested in understanding process variability, mixed models with multiple variance components are used to account for variability due to operators, due to batches of raw material, due to machine differences, due to measurement errors, and so on. In such situations it is of interest to estimate the components of variance and provide lower and upper confidence bounds for them.

Confidence intervals for variance components have been an important research area for more than 70 years. Interestingly, the first published work on interval estimation for the between-groups variance component in the standard one-way normal random model was by R. A. Fisher (1935), who gave a solution to this problem using his then-new method of fiducial argument. Bross (1950) provided further computational details for the fiducial approach and informally compared it with approximate frequentist methods available at the time. Numerous subsequent articles have been published on this topic (see, e.g., Green 1954; Huitson 1955; Graybill, Martin, and Godfrey 1956; Welch 1956; Healy 1961, 1963; Williams 1962; Broemeling 1969; Burdick and Sielken 1978; Venables and James 1978; Graybill and Wang 1980; Jeyaratnam and Graybill 1980; Seely 1980; Burdick and Graybill 1984; Harville and Fenech 1985; Wild 1981, among others). Most of these articles are concerned with developing exact or approximate confidence intervals for specified linear functions of variance components or their ratios. Some of the work was carried out in the context of inference on a heritability coefficient in animal breeding studies. Healy (1963), Venables and James (1978), and Wild (1981) considered fiducial approaches to the problem in the case of balanced data.

Our focus in this article is on unbalanced normal mixed linear models with two variance components. There are several good reasons for limiting ourselves to these models. Two-component mixed models are actually a fairly general class, because no restrictions are placed on the fixed-effects part of the model. In addition, closed-form expressions for minimal sufficient statistics are available for this situation. Such closed-form expressions for minimal sufficient statistics typically are unavailable for general (unbalanced) mixed models with more than two variance components. Although in principle the fiducial approach still can be implemented in these cases, one loses the computational advantages that accompany closed-form expressions for minimal sufficient statistics. These are some of the reasons possibly explaining why most of the publications on this topic address only the special case of two-component mixed models.

Although many works have addressed interval estimation problems for the two–variance component mixed linear model and its various special cases, a fiducial solution to the interval estimation problem in this context is not currently available. Here we develop such a fiducial solution and demonstrate through a simulation study that the resulting procedure has better overall frequentist performance than competing methods. We also establish the asymptotic exactness of the coverage probability of fiducial intervals for variance components of interest. Although we focus on confidence interval estimation, our results can be used to carry out hypothesis tests about the variance components. In the context of recovery of intrablock information, Portnoy (1973) discussed tests of the null hypothesis that the variance component associated with blocks is 0 and proposed improved tests of parameters in such models. The procedures that we develop in this article automatically make use of both interblock and intrablock information.

More specifically, let $\mathbf{Y}$ denote a $N \times 1$ vector of observable random variables. Suppose that $\mathbf{Y}$ has a distribution described

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by the following mixed linear model with two variance components:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \boldsymbol{\varepsilon}, \quad (1)$$

where $\mathbf{X}$ and $\mathbf{Z}$ are known incidence matrixes of sizes $N \times p$ and $N \times a$, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown parameters, $\mathbf{u} \sim \mathrm{N}(\mathbf{0}, \sigma_\alpha^2 \mathbf{A})$ is a $a \times 1$ vector of random effects, $\boldsymbol{\varepsilon} \sim \mathrm{N}(\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_N)$ is the error vector of size $N \times 1$, and $\mathbf{u}$ and $\boldsymbol{\varepsilon}$ are independent. Without loss of generality, we assume that $\mathrm{rank}(\mathbf{X}) = p$. $\mathbf{A}$ is a known matrix often referred to as a *relationship matrix* in animal breeding context, because it describes the degree to which the elements $u_1, \ldots, u_a$ of the vector $\mathbf{u}$ covary. For example, if the elements u_1 and u_2 of $\mathbf{u}$ are the (additive) genetic effects corresponding to a parent and an offspring, then $\mathrm{cov}(u_1, u_2) = \sigma_\alpha^2/2$ (Falconer 1989). Note that the standard unbalanced one-way random model given by

$$Y_{ij} = \mu + u_i + \varepsilon_{ij}, \qquad i = 1, \ldots, a;\ j = 1, \ldots, n_i, \quad (2)$$

is a special case of model (1).

In this article we focus on constructing confidence intervals for the variance components σ_α^2 and σ_ε^2 and the heritability coefficient $\rho = \sigma_\alpha^2/(\sigma_\alpha^2 + \sigma_\varepsilon^2)$. In the special case of a one-way random-effects model, σ_α^2 is the between-groups variance component and ρ is the intraclass correlation coefficient. Our proposed methods follow the fiducial generalized pivotal quantity (FGPQ)-based interval procedures discussed by Hannig, Iyer, and Patterson (2006) and the generalizations of the fiducial method given in Hannig (2008).

The article is organized as follows. Section 2 provides a brief review of published confidence interval procedures for σ_α^2, σ_ε^2, and ρ. Section 3 outlines the fiducial method for obtaining confidence intervals for general situations, then applies this method to derive fiducial confidence intervals for σ_α^2, σ_ε^2, and ρ. Our procedure is applicable to the two-component mixed model given in (1). Finally, our proposed procedures for σ_α^2 are compared with competing methods described in Section 2 using a simulation study. Section 4 provides details of the simulation study, along with a discussion of the simulation results. Section 5 considers some data examples using previously published data and illustrates how our proposed procedures are applied. Finally, Section 6 concludes with summary discussions. Derivations of fiducial densities and proof of the asymptotic exactness of the proposed fiducial intervals are given in the Appendixes.

2. INTERVALS FOR TWO–COMPONENT MIXED MODELS

In this section we list some of the published confidence intervals for σ_α^2, σ_ε^2, and $\rho = \sigma_\alpha^2/(\sigma_\alpha^2 + \sigma_\varepsilon^2)$ in a two-component mixed model, which we compared with the proposed fiducial approach in the simulation study reported in Section 4. First, we briefly review some well-known results concerning minimal sufficient statistics for the mixed model in (1).

Let $\mathbf{H}$ be a $N \times (N - p)$ matrix such that $\mathbf{H}\mathbf{H}^T = \mathbf{I}_N - \mathbf{X}(\mathbf{X}^T\mathbf{X})^- \mathbf{X}^T$ and $\mathbf{H}^T\mathbf{H} = \mathbf{I}_{N-p}$. Using the fact that $\mathbf{Y} \sim \mathrm{N}(\mathbf{X}\boldsymbol{\beta}, \sigma_\varepsilon^2 \mathbf{I}_N + \sigma_\alpha^2 \mathbf{Z}\mathbf{A}\mathbf{Z}^T)$, it follows that

$$\mathbf{H}^T\mathbf{Y} \sim \mathrm{N}(\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_{N-p} + \sigma_\alpha^2 \mathbf{G}), \quad (3)$$

where $\mathbf{G} = \mathbf{H}^T\mathbf{Z}\mathbf{A}\mathbf{Z}^T\mathbf{H}$. Let $\lambda_1 > \cdots > \lambda_d \ge 0$ be the distinct eigenvalues of $\mathbf{G}$ with multiplicities $r_1, \ldots, r_d$. Let $\mathbf{P} = [\mathbf{P}_1, \ldots, \mathbf{P}_d]$ be a $(N - p) \times (N - p)$ orthogonal matrix such that $\mathbf{P}^T\mathbf{G}\mathbf{P} = \mathrm{diag}(\lambda_1 \mathbf{1}_{r_1}^T, \ldots, \lambda_d \mathbf{1}_{r_d}^T)$, where $\mathbf{P}_i$ corresponding to λ_i is of size $(N - p) \times r_i$. Define

$$V_i = \mathbf{Y}^T\mathbf{H}\mathbf{P}_i\mathbf{P}_i^T\mathbf{H}^T\mathbf{Y}, \qquad i = 1, \ldots, d. \quad (4)$$

Olsen, Seely, and Birkes (1976) showed that $(V_1, \ldots, V_d)$ is minimally sufficient for $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ under (3). Furthermore,

$$U_i = \frac{V_i}{\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2} \sim \chi_{r_i}^2, \qquad i = 1, \ldots, d, \quad (5)$$

and the U_i's are mutually independent, where χ_v^2 represents a central chi-squared distribution with v degrees of freedom. Note that when λ_d is 0, a *pure error* estimate of σ_ε^2 is given by V_d/r_d. An exact $100(1-\alpha)\%$ confidence interval for σ_ε^2 exists and is given by

$$\left[\frac{V_d}{\chi_{1-\alpha/2;r_d}^2}, \frac{V_d}{\chi_{\alpha/2;r_d}^2}\right], \quad (6)$$

where $\chi_{\alpha;v}^2$ represents the 100α-percentile of the chi-squared distribution with v degrees of freedom. We refer to the interval in (6) as an EXACT (EX) confidence interval for σ_ε^2. When $\lambda_d > 0$, a pure error estimate of σ_ε^2 is not available. In particular, an exact confidence interval for σ_ε^2 is unavailable.

2.1 Confidence Intervals for σ_α^2 in an Unbalanced One-Way Random-Effects Model

Several methods are available for constructing approximate confidence intervals for σ_α^2 in the unbalanced one-way random-effects model. We used five different confidence interval procedures for σ_α^2 appearing in the literature in our simulation study as competitors to our fiducial approach: (a) the Burdick–Graybill (BG) confidence interval (Burdick and Graybill 1992), (b) the Thomas–Hultquist (TH) confidence interval (Thomas and Hultquist 1978), (c) the Burdick–Eickman (BE) confidence interval (Burdick and Eickman 1986), (d) the Hartung–Knapp (HK) confidence interval (Hartung and Knapp 2000), and (e) the Arendacká (Ar) confidence interval (Arendacká 2005). (For a summary of these methods, see E et al. 2006). Note that the HK method used here is the "better" of the two procedures proposed by Hartung and Knapp (2000). It also is important to note that the first four interval procedures listed here apply only for the one-way random model; they do not apply to the general two-component mixed model in (1). For this case, the Ar method is applicable when a pure error estimate of σ_ε^2 is available.

2.2 Confidence Intervals for σ_ε^2 in a Two–Variance Components Mixed Model

As mentioned earlier, an exact confidence interval for σ_ε^2 is available when $\lambda_d = 0$; that is, a pure error estimate of σ_ε^2 is available. However, for the case where $\lambda_d > 0$, to the best of our knowledge, no confidence interval procedure for σ_ε^2 has yet been proposed. Here we propose a fiducial interval estimate for σ_ε^2 that appears to have satisfactory coverage properties. We discuss the fiducial approach in Section 3.

2.3 Confidence Intervals for ρ in a Two–Variance Component Mixed Model

In many applications, the quantity $\rho = \sigma_\alpha^2/(\sigma_\alpha^2 + \sigma_\varepsilon^2)$ is of interest. For example, in plant and animal breeding, ρ represents the proportion of the total variance that is explainable by additive genetic effects; it often is referred to as the *heritability* of the trait under study.

Many authors have considered the problem of constructing exact confidence intervals for ρ, beginning with Wald (1940, 1947). Other contributors to this problem include Khuri (1981), Seely and El Bassiouni (1983), Verdooren (1988), Lee and Seely (1996), Fenech and Harville (1991), and Burch and Iyer (1997). The main tool used in these works is the fact that independent quadratic forms V_i, $i = 1, \dots, d$, given in (4) are available, with which a pivotal quantity for ρ may be constructed in the form

$$R = \left(\sum_{i\in I^c} \frac{V_i}{1+\rho(\lambda_i - 1)} \Big/ \sum_{i\in I^c} r_i\right) \Big/ \left(\sum_{j\in I} \frac{V_j}{1+\rho(\lambda_j - 1)} \Big/ \sum_{j\in I} r_j\right), \quad (7)$$

where I is any nonempty subset of $\{1, \dots, d\}$. This pivotal quantity has a central F distribution. Burch and Iyer (1997) studied a subset of pivots of the foregoing form that led to locally unbiased intervals for ρ and recommended the use of an optimal interval from this subclass. We refer to their recommended interval as the BI confidence interval. Because nearly all of the exact intervals for ρ proposed in the literature belong to this class (e.g., the Wald intervals), we compare our proposed fiducial interval for ρ with the BI intervals.

3. FIDUCIAL INTERVALS FOR σ_α^2, σ_ε^2, AND ρ

It is worth noting that generalized confidence intervals, such as those proposed by Arendacká (2005), are closely related to fiducial intervals. This connection between generalized inference and fiducial inference was discussed in detail by Hannig et al. (2006), who also provided a recipe for constructing fiducial intervals when $\mathbf{X}$ has a continuous distribution. Hannig (2008) generalized this to arbitrary distributions. The term *generalized fiducial inference* is used to emphasize the fact that the version of fiducial inference discussed by Hannig et al. (2006) and Hannig (2008) is a generalization of R. A. Fisher's fiducial argument.

In this section we describe fiducial interval (FI) procedures for σ_α^2, σ_ε^2, and ρ that are applicable under the general two-component mixed model in (1). The intervals that we propose are obtained using the fiducial method described by Hannig et al. (2006) and Hannig (2008).

3.1 The Fiducial Approach

Let $\mathbf{X}$ be a random vector with a distribution indexed by a (possibly vector) parameter $\xi \in \Xi$. Hannig (2008) defined a generalized fiducial distribution for ξ as follows. Assume that $\mathbf{X}$ has a *structural representation* given by $\mathbf{X} = G(U, \xi)$, where U is a random variable or random vector whose distribution is fully known and free of unknown parameters and G is a jointly measurable function of U and ξ. Let $R(\mathbf{x}, u)$ be a set-valued function defined by $R(\mathbf{x}, u) = \{\xi : \mathbf{x} = G(u, \xi)\}$. The set $\{\xi : \mathbf{x} = G(u, \xi)\}$ may be empty, may consist of a single element, or, when the distribution of $\mathbf{X}$ is not continuous, may consist of more than one element (possibly uncountably many elements). The function $R(\mathbf{X}, U)$ may be viewed as an inverse of the function G. Here G defines u as an implicit function of ξ and $\mathbf{x}$ is considered fixed. Following Hannig (2008), we define a generalized fiducial distribution of ξ as a conditional distribution of

$$R(\mathbf{x}, U^*) \quad \text{given} \quad \{R(\mathbf{x}, U^*) \neq \emptyset\}. \quad (8)$$

Here $\mathbf{x}$ is the observed value of $\mathbf{X}$, and U^* is an independent copy of U.

If the probability $P(R(\mathbf{x}, U^*) \neq \emptyset) = 0$, as in our case, then the conditioning event must be interpreted using equations involving random variables. Therefore, the fiducial distribution of $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ is not unique. A different choice of the conditioning equations will result in a different fiducial distribution for $(\sigma_\alpha^2, \sigma_\varepsilon^2)$. This is related to the well-known Borel paradox described by, for example, Casella and Berger (2002, sec. 4.9.3). We present a particular way of interpreting (8) that seems very intuitively appealing and leads to a fiducial distribution for σ_α^2 and σ_ε^2 with very good statistical properties.

We begin with the statistics $Q_i = V_i/r_i$, $i = 1, \dots, d$, where V_i and r_i are as defined in (4). Note that these are minimally sufficient for $\{\sigma_\alpha^2, \sigma_\varepsilon^2\}$ under the model in (3). When $d = 2$, the relationship between $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ and (Q_1, Q_2) is invertible. This makes fiducial inference for the case $d = 2$ quite straightforward, and thus we do not consider it here. Hereinafter, we assume that $d > 2$, which is the more general and challenging case. We rewrite the expressions in (5) as

$$\begin{aligned} Q_1 &= \frac{(\lambda_1\sigma_\alpha^2 + \sigma_\varepsilon^2)U_1}{r_1}, \\ Q_2 &= \frac{(\lambda_2\sigma_\alpha^2 + \sigma_\varepsilon^2)U_2}{r_2}, \\ &\vdots \\ Q_d &= \frac{(\lambda_d\sigma_\alpha^2 + \sigma_\varepsilon^2)U_d}{r_d}. \end{aligned} \quad (9)$$

Note that (9) provides a structural representation for the observable random vector $\mathbf{Q} = (Q_1, \dots, Q_d)$ in terms of the random vector $\mathbf{U} = (U_1, \dots, U_d)$ whose distribution is completely known. (The U's are independent, with each U_i having a chi-squared distribution with r_i degrees of freedom.) We denote realized values of Q_i and U_i by q_i and u_i, for $i = 1, \dots, d$.

The main idea in interpreting (8) is to randomly pick two equations in (9) and solve for σ_α^2 and σ_ε^2, then plug these solutions for σ_α^2 and σ_ε^2 into the remaining equations and use them for conditioning. This recipe produces a well-defined joint fiducial distribution of $(\sigma_\alpha^2, \sigma_\varepsilon^2)$. As shown in Appendix A, this fiducial density is

$$f(w_1, w_2) = C \cdot g(w_1, w_2), \quad (10)$$

where

$$g(w_1, w_2) = \left(\sum_{i<j} \frac{(\lambda_i - \lambda_j) q_i q_j}{(\lambda_i w_1 + w_2)(\lambda_j w_1 + w_2)}\right) \times \left(\frac{\exp(-(1/2)\sum_{i=1}^d r_i q_i/(\lambda_i w_1 + w_2))}{\prod_{i=1}^d (\lambda_i w_1 + w_2)^{r_i/2}}\right) \times \prod_{i=1}^d I_{\{\lambda_i w_1 + w_2 > 0\}}$$

and

$$C^{-1} = \int_{-\infty}^0 \int_{-\lambda_1 w_1}^\infty g(w_1, w_2)\, dw_2\, dw_1 + \int_0^\infty \int_{-\lambda_d w_1}^\infty g(w_1, w_2)\, dw_2\, dw_1.$$

For future reference, we denote a random variable with density (10) by $(R_{\sigma_\alpha^2}, R_{\sigma_\varepsilon^2})$.

Hannig et al. (2006) outlined a method that can be used to prove that the fiducial distribution for $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ given in (10) leads to asymptotically correct frequentist inference if d is fixed and $r_i \to \infty$. But this is not sufficient for many applications in which we have numerous different eigenvalues with relatively small multiplicities, such as the loin-eye data set discussed in Section 5. Consequently, we have generalized Hannig's earlier theorem (Hannig et al. 2006) by allowing the number of distinct eigenvalues d to take any value between 2 and n. But this requires that the eigenvalues themselves satisfy some natural conditions related to the Fisher's information to have asymptotically correct frequentist inference. The exact conditions are given in Theorem 1, the proof of which is given in Appendix B.

Theorem 1. Write $n = \sum_{i=1}^d r_i$ and assume that the limits

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^d \frac{\lambda_i^k r_i}{(\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2)^2} = m_k \quad \text{for } k = 0, 1, 2$$

are such that the matrix $\Sigma = \binom{m_0\ \ m_1}{m_1\ \ m_2}$ is positive definite. Then the frequentist coverage probability of the $(1-\alpha)$ equal-tailed fiducial interval based on the joint fiducial distribution of $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ approaches the stated value as $n \to \infty$.

Remark 1. It is worth noting that the Fisher information matrix $\mathcal{F}$ for $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ based on $Q_i, i = 1, \ldots, d$, is the 2×2 matrix whose (j,k) element is given by

$$\sum_{i=1}^d \frac{r_i \lambda_i^{j+k-2}}{2(\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2)^2}$$

for $j, k = 1, 2$. Thus the conditions of the theorem state of the requirement that $\frac{1}{n}\mathcal{F}$ converge to a positive definite matrix $\frac{1}{2}\Sigma$ as $n \to \infty$.

Moreover, the proof of Theorem 1 demonstrates that the fiducial distribution as simply Bayesian posteriors satisfies the Bernstein–von Mises theorem. Thus it is asymptotically efficient.

3.2 A Fiducial Confidence Interval for σ_α^2 and σ_ε^2

A fiducial distribution for σ_α^2 can be easily derived from the joint fiducial distribution of $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ in (10) and is given by

$$f_{R_{\sigma_\alpha^2}}(w_1) = \begin{cases} C\int_{-\lambda_1 w_1}^\infty g(w_1, w_2)\, dw_2 & \text{if } w_1 < 0 \\ C\int_{-\lambda_d w_1}^\infty g(w_1, w_2)\, dw_2 & \text{otherwise.} \end{cases}$$

Let $\mathcal{R}_{\sigma_\alpha^2,\gamma}$ be the 100γ-percentile of the fiducial distribution of σ_α^2. Then a two-sided $(1-\alpha)100\%$ fiducial confidence interval for σ_α^2 is given by

$$\left[\max\left(0, \mathcal{R}_{\sigma_\alpha^2,\alpha/2}\right), \max\left(0, \mathcal{R}_{\sigma_\alpha^2,1-\alpha/2}\right)\right].$$

Similarly, it follows that the fiducial distribution of σ_ε^2 is given by

$$f_{R_{\sigma_\varepsilon^2}}(w_2) = \begin{cases} C\int_{-w_2/\lambda_d}^\infty g(w_1, w_2)\, dw_1 & \text{if } w_2 < 0 \text{ and } \lambda_d > 0 \\ C\int_{-w_2/\lambda_1}^\infty g(w_1, w_2)\, dw_1 & \text{if } w_2 > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where C and $g(w_1, w_2)$ are the same as C and $g(w_1, w_2)$ in (10).

Let $\mathcal{R}_{\sigma_\varepsilon^2,\gamma}$ be the 100γ-percentile of the fiducial distribution of σ_ε^2. Then a two-sided $(1-\alpha)100\%$ fiducial confidence interval for σ_ε^2 is given by

$$\left[\max\left(0, \mathcal{R}_{\sigma_\varepsilon^2,\alpha/2}\right), \max\left(0, \mathcal{R}_{\sigma_\varepsilon^2,1-\alpha/2}\right)\right].$$

3.3 A Fiducial Confidence Interval for ρ

A fiducial distribution for ρ can be easily derived from the joint fiducial distribution of $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ in (10). In fact, we obtain the fiducial density for ρ as the density of $R_\rho = R_{\sigma_\alpha^2}/(R_{\sigma_\alpha^2} + R_{\sigma_\varepsilon^2})$ given by

$$f_{R_\rho}(x) = \begin{cases} C\int_{-\infty}^0 g(x, y)\, dy & \text{if } \frac{x}{1-x} < -\frac{1}{\lambda_d} \text{ and } \lambda_d > 0 \\ C\int_0^\infty g(x, y)\, dy & \text{if } \frac{x}{1-x} > -\frac{1}{\lambda_1} \\ 0 & \text{otherwise,} \end{cases}$$

where

$$g(x, y) = \left(\sum_{i<j} \frac{(\lambda_i - \lambda_j) q_i q_j}{((\lambda_i - 1)xy + y)((\lambda_j - 1)xy + y)}\right) \times \left(\frac{(1-x)^{(\sum_{i=1}^d r_i)/2}|y|}{\prod_{i=1}^d ((\lambda_i - 1)xy + y)^{r_i/2}}\right) \times \exp\left(-\frac{1}{2}\sum_{i=1}^d \frac{(1-x) r_i q_i}{(\lambda_i - 1)xy + y}\right) \times \prod_{i=1}^d I_{\{((\lambda_i - 1)xy + y)/(1-x) > 0\}}$$

and

$$C^{-1} = \begin{cases} \int_{-\infty}^{1/(1-\lambda_d)} \int_{-\infty}^{0} g(x,y)\,dy\,dx \\ \quad + \int_{1}^{\infty} \int_{-\infty}^{0} g(x,y)\,dy\,dx \\ \quad + \int_{1/(1-\lambda_1)}^{1} \int_{0}^{\infty} g(x,y)\,dy\,dx, \\ \quad \text{if } \lambda_d > 1 \\ \int_{1}^{\infty} \int_{-\infty}^{0} g(x,y)\,dy\,dx \\ \quad + \int_{1/(1-\lambda_1)}^{1} \int_{0}^{\infty} g(x,y)\,dy\,dx \\ \quad \text{if } \lambda_d = 1 \\ \int_{1}^{1/(1-\lambda_d)} \int_{-\infty}^{0} g(x,y)\,dy\,dx \\ \quad + \int_{-\infty}^{1} \int_{0}^{\infty} g(x,y)\,dy\,dx \\ \quad + \int_{1/(1-\lambda_1)}^{\infty} \int_{0}^{\infty} g(x,y)\,dy\,dx \\ \quad \text{if } 0 < \lambda_1 < 1 \\ \int_{1}^{1/(1-\lambda_d)} \int_{-\infty}^{0} g(x,y)\,dy\,dx \\ \quad + \int_{-\infty}^{1} \int_{0}^{\infty} g(x,y)\,dy\,dx \\ \quad \text{if } \lambda_1 = 1 \\ \int_{1}^{1/(1-\lambda_d)} \int_{-\infty}^{0} g(x,y)\,dy\,dx \\ \quad + \int_{1/(1-\lambda_1)}^{1} \int_{0}^{\infty} g(x,y)\,dy\,dx \\ \quad \text{if } \lambda_1 > 1 \text{ and } 0 \le \lambda_d < 1. \end{cases}$$

Let $\mathcal{R}_{\rho,\gamma}$ be the 100γ-percentile of the fiducial distribution of ρ. Then a two-sided $(1-\alpha)100\%$ fiducial confidence interval for ρ is given by

$$\big[\max\big(0, \min\big(\mathcal{R}_{\rho,\alpha/2}, 1\big)\big), \max\big(0, \min\big(\mathcal{R}_{\rho,1-\alpha/2}, 1\big)\big)\big].$$

The next two sections describe details of simulation studies that we conducted to compare the proposed fiducial interval for σ_α^2, σ_ε^2, and ρ with previously proposed methods.

4. SIMULATION STUDY AND DISCUSSION

In this and subsequent sections, we use the abbreviations introduced in Sections 2 and 3 when referring to various competing procedures. The coverage probability of a confidence interval on σ_α^2 depends on the design (e.g., number of within-group measurements, $n_1, \dots, n_a$) as well as on the values of σ_α^2 and σ_ε^2. The degree of imbalance of the design in the case of a one-way random-effects model has been quantified by Ahrens and Pincus (1981) using the measure Φ, defined as $\Phi = a\tilde{n}/N$ with $N = \sum_{i=1}^{a} n_i$ and $\tilde{n} = a/\sum_{i=1}^{a}(1/n_i)$. Note that $0 < \Phi \le 1$ and that Φ equals 1 if and only if n_i are all equal. The smaller the value of Φ, the greater the degree of imbalance. For our simulation study, we selected seven different unbalanced patterns, as shown in Table 1. Patterns 1, 2, and 5 also were considered by Hartung and Knapp (2000); pattern 4 also was considered by Arendacká (2005). We added the additional patterns 3, 6, and 7 to study the performance of confidence intervals in small-sample situations. Without loss of generality, we assumed that $\mu = 0$. The values selected for $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ were (.1, 10), (.5, 10), (1, 10), (.5, 2), (1, 1), (2, .5), (5, .2), and (10, .1), where the settings (.1, 10), (.5, 2), (1, 1), (2, .5), (5, .2) were used by Arendacká (2005). We added three more settings to our study to better investigate the performance of confidence intervals under extremely large and small values of the ratio $\sigma_\alpha^2/\sigma_\varepsilon^2$.

Table 1. Unbalanced patterns used in the simulation study

Pattern	Φ	a	n_i					
1	.068	6	1	1	1	1	1	100
2	.130	6	2	2	2	2	2	100
3	.187	3	2	5	60			
4	.410	5	4	4	4	8	48	
5	.700	6	5	10	15	20	25	30
6	.807	4	2	2	4	6		
7	.957	6	6	6	8	8	10	10

For each setting of sample size n_i and values of $(\sigma_\alpha^2, \sigma_\varepsilon^2)$, we generated 3,000 independent data sets and computed two-sided 95% confidence intervals for σ_α^2 for each method. We compared the (a) BG interval, (b) TH interval, (c) BE interval, (d) HK interval, (e) Ar interval, and (f) FI interval. The criteria for judging the performance of the methods were the empirical coverage probabilities and the average lengths of the confidence intervals. The simulation study was programmed in Fortran. Two IMSL subroutines, DQ2AGI and DTWODQ (IMSL 1994), were used to compute the necessary one-dimensional and two-dimensional integrals.

The results of our simulation study are graphically summarized in Figures 1, 2, 3, and 4. Figures 1 and 2 show the empirical coverage probabilities for settings with ratio $\eta = \sigma_\alpha^2/\sigma_\varepsilon^2 < 1$

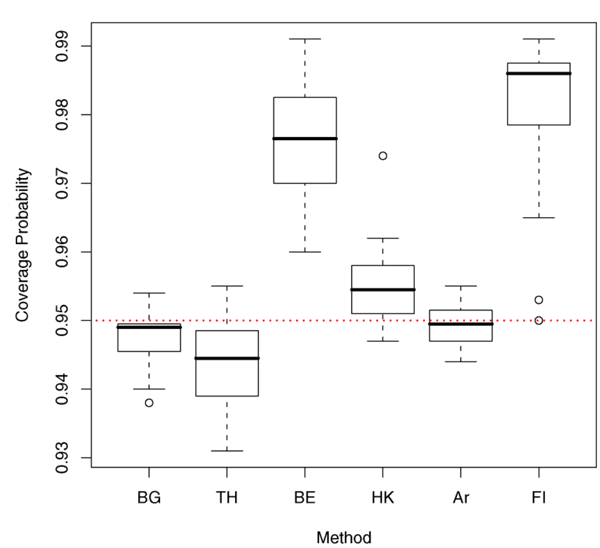

Figure 1. Empirical coverage probabilities for settings with $\eta < 1$.

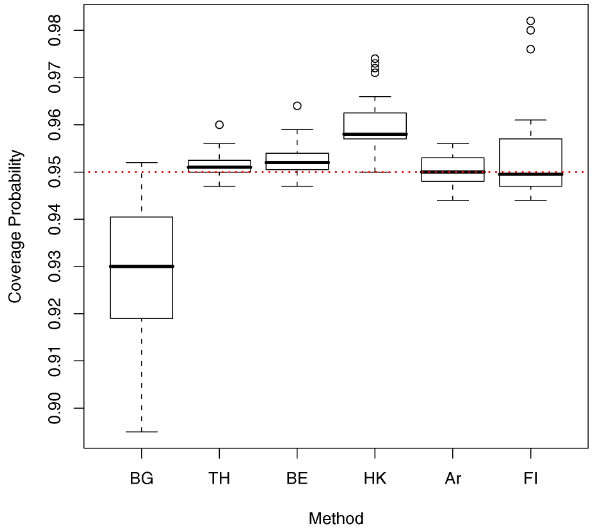

Figure 2. Empirical coverage probabilities for settings with $\eta \geq 1$.

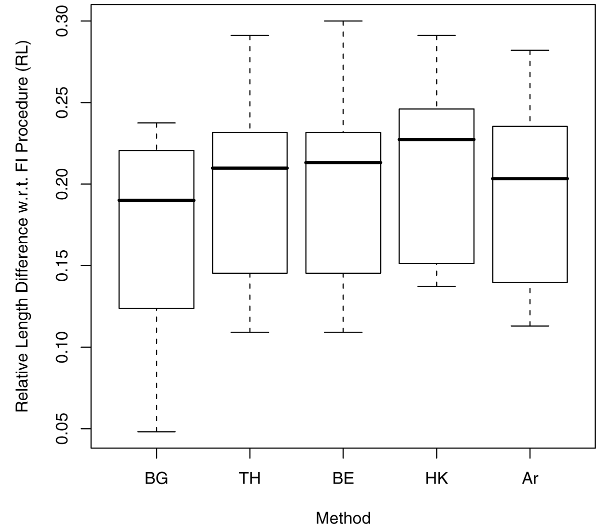

Figure 4. Relative difference of the average confidence interval length (RL) for settings with $\eta \geq 1$.

and for settings with $\eta \geq 1$. Figures 3 and 4 show the differences of the average confidence interval lengths, relative to the fiducial interval, for all competing procedures for settings with $\eta < 1$ and settings with $\eta \geq 1$. These relative lengths are denoted by *RL*, which is defined as $(L_M - L_{FI})/L_{FI}$, where L_M denotes the average length of a competing interval and L_{FI} denotes the average length of the FI interval. (For the detailed numerical simulation results, see E et al. 2006.)

The results show that BG procedure is very liberal when the ratio η is large. The TH procedure is liberal for small values of η and very unbalanced designs. This finding agrees with the findings of Burdick and Eickman (1986). The BE procedure is conservative, and its behavior for large η is similar to that of the TH procedure. The HK procedure becomes more conservative as the value of η becomes large. The Ar procedure appears to always maintain the stated confidence coefficient. The FI interval is conservative when the ratio $\eta < 1$ but maintains the stated confidence coefficient when $\eta \geq 1$.

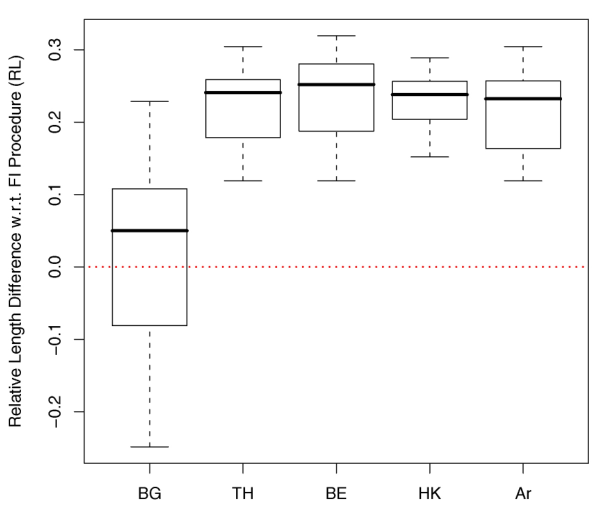

Figure 3. Relative difference of the average confidence interval length (RL) for settings with $\eta < 1$.

Comparing average interval lengths, we see that all of the intervals behave very similarly except the BG and FI intervals. Although the BG interval has small average lengths, it does not adequately maintain the stated coverage probabilities when η is large; therefore, the BG interval is not recommended. Compared with procedures other than the BG procedure, the FI interval always has the smallest average lengths and standard deviations, even when it is conservative. The average lengths of FI intervals are 10–25% smaller than the average lengths of all other intervals except the BG interval. Based on these results, we recommend the FI intervals for σ_α^2 as the most suitable choice for practical applications.

5. EXAMPLES

As noted earlier, a fiducial interval for σ_α^2, σ_ε^2, and ρ is available in the general mixed model (1) with two variance components. In this section we give two examples, one involving incomplete-block designs for slope-ratio assays and the other arising from animal breeding studies. In the first example is, from Das and Kulkarni (1966), the degrees of freedom for error is positive and the eigenvalue λ_d is 0. The second example uses a model that may be referred to as a *full animal model*. All eigenvalues λ_j, $j = 1, \ldots, d$, are positive, and thus there are no degrees of freedom available for error.

5.1 Incomplete-Block Design for the Slope-Ratio Assay

In a $(2k+1)$-point symmetrical slope-ratio assay, equal numbers of subjects are given each of k standard and test preparations and a blank dose. The responses are assumed to linearly depend on dose, usually on a logarithmic scale. This $(2k+1)$-point symmetrical slope-ratio assay requires blocks of size $2k+1$ for a randomized complete-block design. Das and Kulkarni (1966) developed a modified Balanced Incomplete Block (BIB) design with blocks of size $2k'+1$ $(k' < k)$ for slope-ratio assays. Suppose that s_i and t_i, $i = 1, \ldots, k$, are the ith dose levels of standard preparation and test preparation, where doses are equally spaced and sorted in ascending order. First, a BIB design for k doses of the standard preparation in blocks of size k' is obtained and used as the basic design. Then the modified BIB design is obtained by augmenting every block of the basic BIB design by a blank dose and k' doses of the test preparation, using the rule that dose t_i should be included in every block containing dose s_i. Das and Kulkarni (1966) claimed that the modified BIB design is more efficient than the randomized complete-block design. Kulshreshtha (1969) later proved that the new design gives shorter confidence intervals for relative potency based on Fieller's theorem than the random block design with equal replication of nonzero doses. The relative potency is defined as the ratio of the slope of the dose-response curve for the test preparation to that for the standard preparation. The model for slope-ratio assay considered by Das and Kulkarni (1966) and Kulshreshtha (1969) can be described by

$$y_{ijm} = \mu + \beta_i x_{ij} + \gamma_m + \epsilon_{ijm},$$
$$i = s, t, \text{ or } c;\ j = 1, \ldots, k,\ m = 1, \ldots, b, \quad (11)$$

where y_{sjm}, y_{tjm}, and y_{cjm} denote the observation in mth block for jth dose of standard preparation, test preparation, and blank dose; x_{sj} and x_{tj} denote the jth dose of the standard and test preparation; x_{cj} is equal to 0; γ_k represents the effect of kth block; and ϵ_{ijm} are iid random measurement errors with a $N(0, \sigma_\varepsilon^2)$ distribution. The block effect γ_m was taken to be fixed by Das and Kulkarni (1966) and Kulshreshtha (1969). To illustrate the methods of this article, we consider blocks to be random and assume that $\gamma_m \overset{\text{iid}}{\sim} N(0, \sigma_\alpha^2)$. Furthermore, we assume that γ_m are independent of ϵ_{ijm}.

Das and Kulkarni (1966) gave several real data examples to illustrate the construction and analysis of the new designs. One example is a nine-point slope-ratio assay on riboflavin content of yeast, with two replications of each dose. These data were first used by Bliss (1952). Das and Kulkarni (1966) deleted the observations on the highest dose of each preparation and used the remaining data to develop a modified BIB design for seven doses in three blocks of size five, with two replications of each preparation. The observations of titer per tube, arranged according to this design, are given in Table 2. Here we calculate the fiducial distributions associated with σ_α^2, σ_ϵ^2, and ρ.

There are three distinct eigenvalues of $\mathbf{G} = \mathbf{H}^T\mathbf{ZAZ}^T\mathbf{H}$: $\lambda_1 = 5$ with multiplicity $r_1 = 1$, $\lambda_2 = 4.545455$ with multiplicity $r_2 = 1$, and $\lambda_3 = 0$ with multiplicity $r_3 = 10$. The method-of-moments (MOM) estimates of σ_α^2 and σ_ε^2 are .0033 and .1045. The corresponding estimate of ρ is .0306. The restricted maximum likelihood (REML) estimates of σ_α^2 and σ_ε^2 are the same as the MOM estimates.

Figures 5, 6, and 7 show plots of the fiducial densities of σ_α^2, σ_ε^2, and ρ. Note that the support of the fiducial density for σ_α^2, σ_ε^2, and ρ might be a proper superset of their natural boundaries. For instance, observe that the fiducial density for ρ for

Table 2. Data and modified BIB design for the example of the slope-ratio assay

Block	Blank	Standard			Test		
	c	s_1	s_2	s_3	t_1	t_2	t_3
1	.72	2.15	4.35		2.35	4.40	
2	.78		4.05	6.10		4.70	6.10
3	.76	2.30		5.60	2.45		5.10

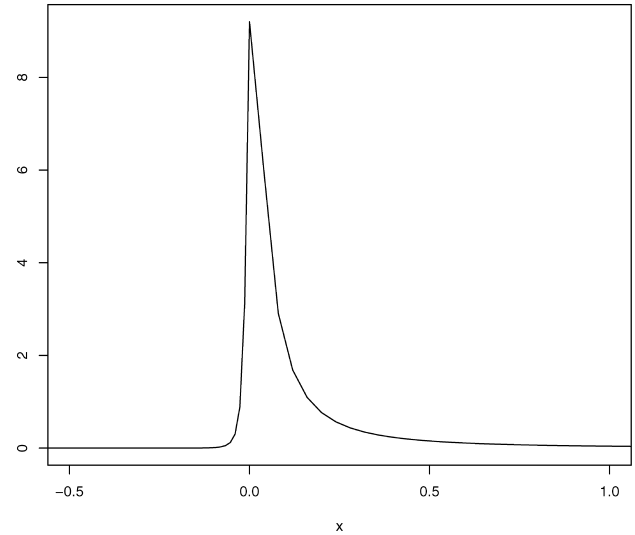

Figure 5. Fiducial density plot for σ_α^2 for the slope-ratio assay data.

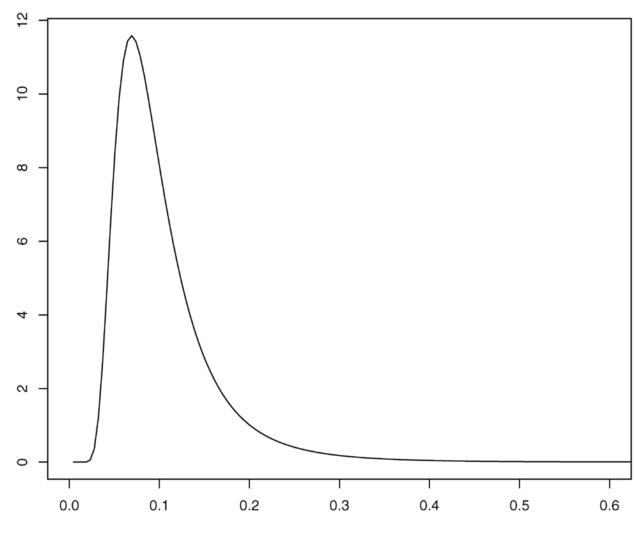

Figure 6. Fiducial density plot for σ_ε^2 for the slope-ratio assay data.

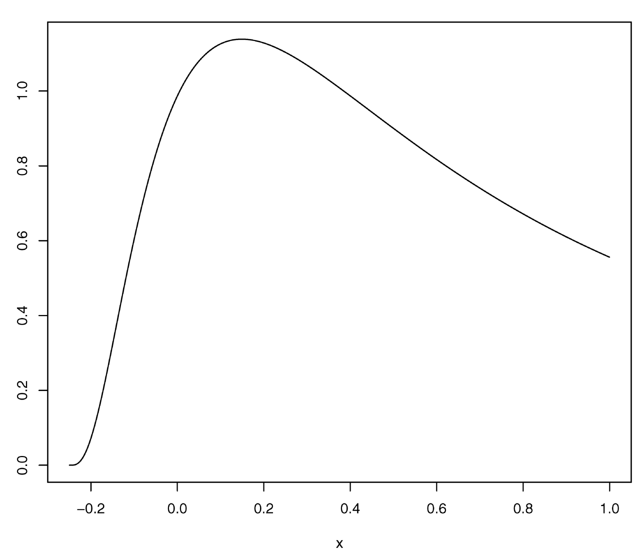

Figure 7. Fiducial density plot for ρ for the slope-ratio assay data.

Table 3. Nominal 90% and 95% confidence intervals on σ_α^2 for the slope-ratio assay data

Method	90%	95%
Ar	(0, .898)	(0, 1.841)
FI	(0, .875)	(0, 1.781)

this data has a range of ρ equal to the interval $(1/(1-\lambda_1), 1)$, that is, $(-.25, 1)$. When calculating fiducial confidence intervals, we replace negative confidence bounds with 0, and when a confidence bound for ρ happens to be larger than 1, we replace it with 1. Table 3 gives the Ar and the FI confidence intervals for σ_α^2 with 90% and 95% nominal confidence coefficients.

In this example, it might be of interest to test the existence of the block random effect, that is, the hypothesis of $H_0 : \sigma_\alpha^2 = 0$ versus $H_a : \sigma_\alpha^2 > 0$. Portnoy (1973) proposed an efficient test of the foregoing hypothesis, that used both intrablock (i.e., between-subjects) and interblock (i.e., within-subjects) information. The test is based on three independent scaled chi-squared statistics,

$$T \sim (\sigma_\varepsilon^2 + a\sigma_\alpha^2)\chi_{n_1}^2,$$

$$S_1 \sim (\sigma_\varepsilon^2 + b\sigma_\alpha^2)\chi_{n_2}^2, \qquad \text{and} \qquad S_2 \sim \sigma_\varepsilon^2 \chi_m^2.$$

The null hypothesis is rejected if

$$\frac{(S_1 + T)/(n_1 + n_2)}{S_2/m} > F_{1-\alpha;(n_1+n_2),m}, \tag{12}$$

where $F_{\gamma;v_1,v_2}$ represents the γ-quantile of the F-distribution with v_1 and v_2 degrees of freedom. Portnoy's test statistic calculated from this slope-ratio assay data are equal to 2.7930, less than $F_{.95;2,10} = 4.1028$. Thus we are unable to reject H_0. Note that the test given in (12) cannot be inverted to provide a confidence interval of σ_α^2, because the test is applicable for testing the hypothesis $H_0 : \sigma_\alpha^2 = \sigma_0^2$ for the special case where $\sigma_0^2 = 0$. On the other hand, the fiducial approach proposed here can be used to obtain a confidence interval for σ_α^2.

The hypothesis $\sigma_\alpha^2 = 0$ also can be tested using the fiducial confidence interval procedure. In particular, for this example, the 95% one-sided fiducial interval for σ_α is $(-.0095, \infty)$, which contains 0. We again fail to reject H_0. Thus in this example, the test of Portnoy (1973) and the test based on a fiducial interval both reach the same conclusion.

For sake of completeness, Table 4 shows the EX and the FI confidence intervals for σ_ε^2 with 90% and 95% nominal confidence coefficients. For this example, there does not exist an unbiased BI confidence interval for ρ. In this case we take $I = \{1, 2\}$ in (7), which gives us the pivotal quantity with the closest "balance" between the numerator and denominator degrees of freedom, where $r_3 = 10$ and $\sum_{i=1}^2 r_i = 2$. Table 5 gives the FI and BI confidence intervals for ρ with 90% and 95% nominal confidence coefficients.

Table 4. Nominal 90% and 95% confidence intervals on σ_ε^2 for the slope-ratio assay data

Method	90%	95%
EX	(.045, .210)	(.040, .254)
FI	(.045, .211)	(.040, .257)

Table 5. Nominal 90% and 95% confidence intervals on ρ for the slope-ratio assay data

Method	90%	95%
BI	(0, .913)	(0, .956)
FI	(0, .916)	(0, .957)

5.2 Full Animal Model

These data were used previously by Burch (1996) and Burch and Iyer (1997). Data were obtained on 171 yearling bulls from a red Angus seed stock in Montana. A trait of interest was the loin eye (i.e., ribeye) muscle area, measured in square inches. Ultrasound techniques were used to obtain these measurements. The fixed effect was age of the dam, in one of five categories: 2 years, 3 years, 4 years, 5–9 years, or 10 or more years. The random effects are the animal's (additive) genetic effect and error. The mixed linear model under consideration can be represented by

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \boldsymbol{\varepsilon},$$

where $\mathbf{Y}$ is a 171×1 vector of observable random variables, $\mathbf{X}$ is a 171×5 design matrix, $\boldsymbol{\beta}$ is a 5×1 vector of unknown parameters, $\mathbf{Z} = \mathbf{I}_{171}$, and $\mathbf{u}$ and $\boldsymbol{\varepsilon}$ are vectors of unobservable random variables of size 171×1. The relationship matrix $\mathbf{A}$ was determined using a recursive method given by Henderson (1976); this means that $\mathrm{var}(\mathbf{u}) = \sigma_\alpha^2 \mathbf{A}$. The number of distinct eigenvalues of $\mathbf{G} = \mathbf{H}^T \mathbf{Z}\mathbf{A}\mathbf{Z}^T \mathbf{H}$ was $d = 165$. Eigenvalues ranged in magnitude from $\lambda_1 = 8.5692472$ to $\lambda_{165} = .5656916$. Except for $\lambda_{105} = .6718750$, with $r_{105} = 2$, all eigenvalues had a multiplicity of 1. The REML estimates of σ_α^2 and σ_ε^2 were .2994 and 2.6539. The corresponding estimate of ρ was .1014. We refer to this estimate the as REML estimate of ρ.

Figures 8, 9, and 10 show plots of the fiducial densities for σ_α^2, σ_ε^2, and ρ for the loin-eye data. The support of the fiducial density for σ_α^2 and for σ_ε^2 is $(-\infty, \infty)$. The support of the fiducial density for ρ is

$$\left\{\rho : \rho \in \left(\frac{1}{1-\lambda_1}, 1\right) \cup \left(1, \frac{1}{1-\lambda_d}\right)\right\},$$

that is, $\{\rho : \rho \in (-.1321, 1) \cup (1, 2.3025)\}$. The FI confidence intervals for σ_α^2 with 90% and 95% nominal confidence coefficients are (0, 3.000) and (0, 3.750). The FI confidence intervals

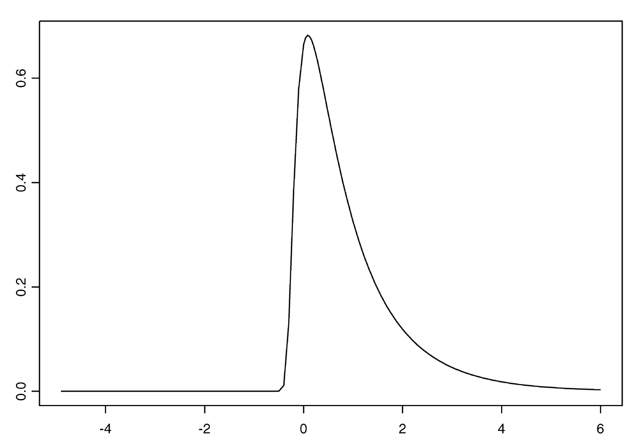

Figure 8. Fiducial density plot for σ_α^2 for the loin-eye data.

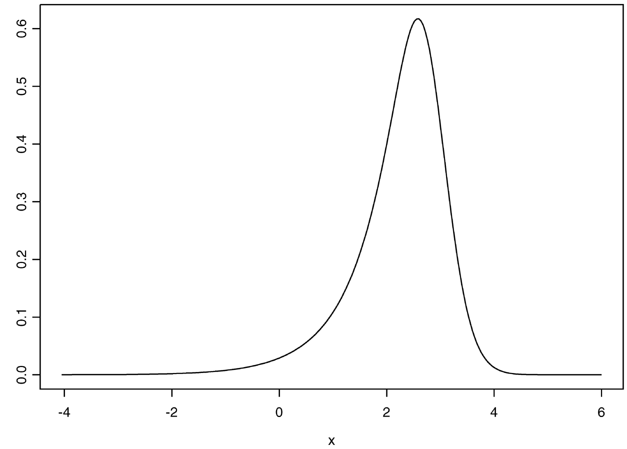

Figure 9. Fiducial density plot for σ_ε^2 for the loin-eye data.

for σ_ε^2 with 90% and 95% nominal confidence coefficients are $(.625, 3.341)$ and $(.100, 3.513)$.

We estimated the coverage probabilities corresponding to the nominally 90% and 95% two-sided FI confidence intervals on σ_α^2 and σ_ε^2 using simulation with REML estimates of σ_α^2 and σ_ε^2 as their true values. The results are based on 2,000 generated independent data sets. The simulation estimates of the empirical coverages for FI intervals on σ_α^2 are .935 and .975, corresponding to nominal confidence coefficients of .90 and .95. For the FI intervals on σ_ε^2 the coverage probability estimates are .923 and .959, corresponding to nominal confidence coefficients of .90 and .95.

The BI pivotal quantity that results in a locally unbiased confidence interval corresponds to $I = \{1, \ldots, 83\}$ in (7). In this case, $\sum_{i=1}^{83} r_i = \sum_{j=84}^{165} r_j = 83$. We refer to this unbiased confidence interval as the BI confidence interval in what follows. Table 6 gives the FI and BI confidence intervals for ρ with 90% and 95% nominal confidence coefficients. It is interesting to see that the BI confidence interval covers the entire parameter space. Inverting the pivotal quantity in (7) results in a confidence interval whose endpoints fall outside of the parameter space. Harville and Fenech (1985) attributed this to a lack of sufficient information in the data about the parameter of interest. Table 7 gives the empirical coverages of these interval procedures for ρ using REML estimates of σ_α^2, σ_ε^2, and ρ as their true values. The results show that the FI method leads to a shorter confidence interval for ρ in this data set. Comparing the empirical coverages shows that the FI confidence interval is more conservative than the BI confidence interval. In summary, the FI method performs better than the BI method for this data set.

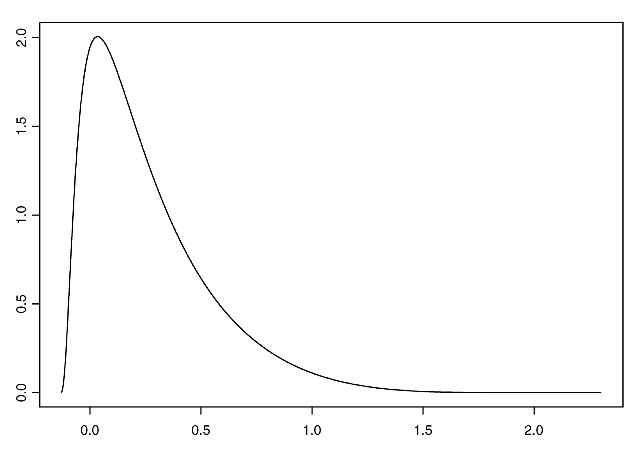

Figure 10. Fiducial density plot for ρ for the loin-eye data.

Table 6. Nominally 90% and 95% confidence intervals on ρ for the loin-eye data

Method	90%	95%
BI	(0, 1.000)	(0, 1.000)
FI	(0, .824)	(0, .972)

6. CLOSING REMARKS

In this article, we have proposed interval estimation procedures for σ_α^2, σ_ε^2, and ρ in a two-component mixed-effects linear model using the fiducial approach. We reported a simulation study carried out to compare the proposed confidence interval for σ_α^2 with five other confidence intervals from the literature, the proposed confidence interval for σ_ε^2 with an exact confidence interval, and the proposed confidence interval for ρ with the method due to Burch and Iyer (1997). The results of a simulation study showed that the proposed fiducial intervals for σ_α^2 are satisfactory in terms of coverage probability. Although they are conservative for small values of the variance ratio $\eta = \sigma_\alpha^2/\sigma_\varepsilon^2$, they have the smallest average interval lengths among all confidence intervals. We gave two examples to illustrate the use of the proposed procedures. The results confirm that the fiducial intervals can be recommended for practical use instead of the methods previously discussed in the literature. The code implementing the proposed method is available from the authors on request.

APPENDIX A: DERIVATION OF THE FIDUCIAL DENSITY

As mentioned earlier, we interpret the fiducial distribution (8) as follows. Pick randomly two equations in (9) and solve for σ_α^2 and σ_ε^2. Then plug these solutions for σ_α^2 and σ_ε^2 into the remaining equations and use them for conditioning.

More formally, the set-valued function $R(\mathbf{q}, \mathbf{U}^\star)$ in (8) is the set of all σ_α^2 and σ_ε^2, with $\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2 > 0$, $i = 1, \ldots, d$, for which

$$q_i = \frac{(\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2)U_i^\star}{r_i}, \qquad i = 1, \ldots, d, \tag{A.1}$$

is satisfied. Here $\mathbf{U}^\star$ is an independent copy of $\mathbf{U}$. In particular, assuming that equations i, j in (A.1) were chosen and fixed, we solve them

Table 7. Empirical coverages of the nominally 90% and 95% two-sided confidence intervals on ρ for the loin-eye data using REML estimates of σ_α^2, σ_ε^2, and ρ as their true values (based on 2,000 simulations)

Method	90%	95%
BI	.900	.951
FI	.939	.977

for σ_α^2 and σ_ε^2. This gives

$$\sigma_\alpha^2 = \frac{1}{(\lambda_i-\lambda_j)}\left(\frac{r_iq_i}{U_i^\star}-\frac{r_jq_j}{U_j^\star}\right) \quad \text{and} \quad \sigma_\varepsilon^2 = \frac{1}{(\lambda_i-\lambda_j)}\left(-\frac{\lambda_jr_iq_i}{U_i^\star}+\frac{\lambda_ir_jq_j}{U_j^\star}\right). \tag{A.2}$$

The system of equations in (A.1) then has a solution if and only if the values of σ_α^2 and σ_ε^2 in (A.2) also satisfy the remaining equations in (A.1). This requirement leads to the following set of constraints that must be satisfied by $\mathbf{U}^\star$:

$$q_k = \frac{U_k^\star}{r_k(\lambda_i-\lambda_j)}\left(\frac{r_iq_i(\lambda_k-\lambda_j)}{U_i^\star}-\frac{r_jq_j(\lambda_k-\lambda_i)}{U_j^\star}\right) \quad \text{for } k\neq i,j. \tag{A.3}$$

Summarizing, the set $R(\mathbf{q},\mathbf{U}^\star)$ is nonempty if and only if (A.3) holds, in which case the set

$$R(\mathbf{q},\mathbf{U}^\star) = \left\{\left(\frac{1}{(\lambda_i-\lambda_j)}\left(\frac{r_iq_i}{U_i^\star}-\frac{r_jq_j}{U_j^\star}\right), \frac{1}{(\lambda_i-\lambda_j)}\left(-\frac{\lambda_jr_iq_i}{U_i^\star}+\frac{\lambda_ir_jq_j}{U_j^\star}\right)\right)\right\}.$$

This leads us to define the random variables $A_{i,j}$, $S_{i,j}$, and $W_{k,i,j}$ as

$$A_{i,j} = \frac{1}{(\lambda_i-\lambda_j)}\left(\frac{r_iq_i}{U_i^\star}-\frac{r_jq_j}{U_j^\star}\right),$$

$$S_{i,j} = \frac{1}{(\lambda_i-\lambda_j)}\left(-\frac{\lambda_jr_iq_i}{U_i^\star}+\frac{\lambda_ir_jq_j}{U_j^\star}\right),$$

and

$$W_{k,i,j} = \frac{U_k^\star}{r_k(\lambda_i-\lambda_j)}\left(\frac{r_iq_i(\lambda_k-\lambda_j)}{U_i^\star}-\frac{r_jq_j(\lambda_k-\lambda_i)}{U_j^\star}\right).$$

We now can interpret the conditional distribution in (8) as

$$A_{i,j}, S_{i,j} \mid W_{k,i,j} = q_k, \qquad k\neq i,j. \tag{A.4}$$

This conditional distribution has a density proportional to the joint density of $A_{i,j}$, $S_{i,j}$, and W_k, $k\neq i,j$, computed at the points a, s, and $\mathbf{q}$. Routine calculation shows that this density is given by

$$f_{i,j}(a,s,\mathbf{q}) = \frac{(\lambda_i-\lambda_j)q_iq_j}{2^{\sum_{k=1}^d r_k/2}(\lambda_ia+s)(\lambda_ja+s)}\exp\left[-\frac{1}{2}\sum_{k=1}^d\frac{r_kq_k}{\lambda_ka+s}\right] \times\prod_{k=1}^d\frac{r_k^{r_k/2}q_k^{r_k/2-1}}{\Gamma(r_k/2)(\lambda_ka+s)^{r_k/2}}I_{\{\lambda_ka+s>0\}}.$$

Unfortunately, a careful inspection of $f_{i,j}(a,s,\mathbf{q})$ reveals that the conditional distribution (A.4) depends on the arbitrary choice of i,j.

To remedy this nonuniqueness, we have considered the equation i,j as being selected at random. Taking this into account, the fiducial density of $(\sigma_\alpha^2,\sigma_\varepsilon^2)$ in (8) thus can be computed as

$$f(a,s) = \lim_{\varepsilon\to0^+}\left[\left(\binom{d}{2}^{-1}\sum_{i<j}\varepsilon^{-d}P\big(A_{i,j}\in(a,a+\varepsilon), S_{i,j}\in(s,s+\varepsilon), W_{k,i,j}\in(q_k,q_k+\varepsilon), k\neq i,j\big)\right) \Big/ \left(\binom{d}{2}^{-1}\times\sum_{i<j}\varepsilon^{-d+2}P(W_{k,i,j}\in(q_k,q_k+\varepsilon),k\neq i,j)\right)\right] \tag{A.5}$$

Notice that each term of the sum in the numerator of (A.5) converges to $f_{i,j}(a,s,\mathbf{q})$. The limit in (A.5) is then

$$f(a,s) = \frac{\sum_{i<j}f_{i,j}(a,s,\mathbf{q})}{\sum_{i<j}\iint f_{i,j}(a,s,\mathbf{q})\,da\,ds},$$

which simplifies to (10) with $w_1=a$ and $w_2=s$. The derivation is now complete.

APPENDIX B: PROOF OF THEOREM 1

We use the ideas presented in the proof of theorem 1 of Hannig (2008). Define the random vectors

$$\mathbb{S} = \left(\sum_{i=1}^d\frac{r_iQ_i}{(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)^2}, \sum_{i=1}^d\frac{\lambda_ir_iQ_i}{(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)^2}\right)$$

and

$$\mathbf{t} = \left(\sum_{i=1}^d\frac{r_i}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2}, \sum_{i=1}^d\frac{\lambda_ir_i}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2}\right).$$

We show that $(\mathbb{S}-\mathbf{t})/\sqrt{n}$ converges in distribution to a normal random vector.

Toward this end, assume without loss of generality that $r_i=1$ for all i, possibly repeating some eigenvalues several times. We then can write

$$\mathbb{S}-\mathbf{t} = \left(\sum_{i=1}^n\frac{(U_i-1)}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2}, \sum_{i=1}^n\frac{\lambda_i(U_i-1)}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2}\right),$$

where U_i are iid chi-squared random variables with 1 degree of freedom. To prove the convergence, we use the Cramèr–Wold device. Fix a and b and denote

$$c = \max_{j=1,\dots,n}\frac{(a+b\lambda_j)^2}{(\lambda_j\sigma_\alpha^2+\sigma_\varepsilon^2)^2}.$$

By our assumptions, $c/n\to0$. Next, we verify the Lindeberg–Feller condition,

$$\lim_{n\to\infty}\sum_{i=1}^nE\left[\frac{(a+b\lambda_i)^2(U_i-1)^2}{n(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)^2}; \sum_{j=1}^n\frac{(a+b\lambda_j)^2}{n(\lambda_j\sigma_\alpha^2+\sigma_\varepsilon^2)^2}\varepsilon<\frac{(a+b\lambda_i)^2(U_i-1)^2}{n(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)^2}\right]$$

$$\le\lim_{n\to\infty}E\left[c(U_i-1)^2;(a^2m_0+2abm_1+b^2m_2)\frac{\varepsilon}{2}<c(U_i-1)^2\right]$$

$$=0.$$

Thus we conclude that $(\mathbb{S}-\mathbf{t})/\sqrt{n}\xrightarrow{\mathcal{D}}\mathbf{H}=(H_1,H_2)\sim\mathrm{N}(0,2\Sigma)$. By Skorokhod's representation theorem (Billingsley 1995), this convergence can be taken a.s. We assume the a.s. convergence in the rest of the proof.

We now investigate the distribution of $\sqrt{n}(\mathcal{R}_{(\sigma_\alpha^2,\sigma_\varepsilon^2)}-(\sigma_\alpha^2,\sigma_\varepsilon^2))$, where $\mathcal{R}_{(\sigma_\alpha^2,\sigma_\varepsilon^2)}$ denotes a random vector with the distribution described in (10). The density of this random variable is a constant multiple of $r(z_1,z_2)=\binom{d}{2}^{-1}g(\sigma_\alpha^2+z_1/\sqrt{n},\sigma_\varepsilon^2+z_2/\sqrt{n})$, where g is as defined in (10). [For future reference, denote this constant as C_n; that is, the density is $C_n^{-1}r(z_1,z_2)$.] Set $w_1=\sigma_\alpha^2+z_1/\sqrt{n}$ and $w_2=\sigma_\varepsilon^2+z_2/\sqrt{n}$ and consider

$$\log r(z_1,z_2) = -\frac{1}{2}\sum_{i=1}^dr_i\left(\frac{q_i}{\lambda_iw_1+w_2}+\log(\lambda_iw_1+w_2)\right)$$

$$+\log\left(\binom{d}{2}^{-1}\sum_{i<j}\frac{(\lambda_i-\lambda_j)q_iq_j}{(\lambda_i w_1+w_2)(\lambda_j w_1+w_2)}\right). \quad (A.6)$$

Applying Taylor series to each term of the first sum in (A.6), we get

$$\sum_{i=1}^{d} r_i\left(\frac{q_i}{\lambda_i w_1+w_2}+\log(\lambda_i w_1+w_2)\right)$$
$$=-n^{-1/2}z_1\sum_{i=1}^{d}\left(\frac{r_iq_i}{(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)^2}-\frac{r_i}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2}\right)$$
$$-n^{-1/2}z_2\sum_{i=1}^{d}\left(\frac{\lambda_i r_iq_i}{(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)^2}-\frac{\lambda_i r_i}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2}\right)$$
$$+n^{-1}z_1^2\sum_{i=1}^{d}\left(\frac{r_iq_i}{(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)^3}-\frac{r_i}{2(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)^2}\right)$$
$$+n^{-1}2z_1z_2\sum_{i=1}^{d}\left(\frac{\lambda_i r_iq_i}{(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)^3}-\frac{\lambda_i r_i}{2(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)^2}\right)$$
$$+n^{-1}z_2^2\sum_{i=1}^{d}\left(\frac{\lambda_i^2 r_iq_i}{(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)^3}-\frac{\lambda_i^2 r_i}{2(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)^2}\right)$$
$$+\sum_{i=1}^{d} r_i\left(\frac{q_i}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2}+\log(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)\right)+o_{as}(1). \quad (A.7)$$

As noted earlier, the first two terms on the right side of (A.7) converge a.s. as $n\to\infty$ to $-z_1H_1-z_2H_2$. By Slutsky's theorem, the next three terms converge a.s. to $z_1^2m_0+2z_1z_2m_1+z_2^2m_2$. Similarly, set

$$L_n=\binom{d}{2}^{-1}\sum_{i<j}\frac{(\lambda_i-\lambda_j)q_iq_j}{(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)(\lambda_j\sigma_\alpha^2+\sigma_\varepsilon^2)}$$

and note that

$$\log\left(\binom{d}{2}^{-1}\sum_{i<j}\frac{(\lambda_i-\lambda_j)q_iq_j}{(\lambda_i w_1+w_2)(\lambda_j w_1+w_2)}\right)-\log(L_n)\to 0 \quad \text{a.s.}$$

Define

$$K_n=\exp\left(\frac{1}{2}\sum_{i=1}^{d} r_i\left(\frac{q_i}{\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2}+\log(\lambda_i\sigma_\alpha^2+\sigma_\varepsilon^2)\right)-\frac{1}{4}\mathbf{H}^{\mathrm{T}}\Sigma^{-1}\mathbf{H}\right)\Big/\left(2\pi L_n\sqrt{\det(2\Sigma^{-1})}\right)$$

and note that

$$h(z_1,z_2)=\lim_{n\to\infty}K_n r(z_1,z_2)=K\exp\left\{-\frac{1}{4}(z_1^2m_0+2z_1z_2m_1+z_2^2m_2-2z_1H_1-2z_2H_2)\right\} \quad \text{a.s.}$$

Here the constant K is chosen so that $h(z_1,z_2)$ integrates to 1. Note that, conditionally on $\mathbf{H}$, $h(z_1,z_2)$ is a density of a multivariate normal distribution $\mathrm{N}(\Sigma^{-1}\mathbf{H},2\Sigma^{-1})$. Also note that, unconditionally, $\Sigma^{-1}\mathbf{H}\sim\mathrm{N}(0,2\Sigma^{-1})$.

Recall that the density of $\sqrt{n}(\mathcal{R}_{(\sigma_\alpha^2,\sigma_\varepsilon^2)}-(\sigma_\alpha^2,\sigma_\varepsilon^2))$ is $C_n^{-1}r(z_1,z_2)$. Furthermore, the functions $\sqrt{\det(2\Sigma^{-1})}K_n r(\sqrt{2}\Sigma^{-1/2}\mathbf{z}+\Sigma^{-1}\mathbf{H})$ are dominated by $C(1+z_1^2+z_2^2)^{-1}$ for C sufficiently large. Thus the Lebesgue-dominated convergence theorem, the fact that densities integrate to 1, and Fatou's lemma imply that $K_nC_n\to 1$. We conclude that the density $C_n^{-1}r(z_1,z_2)$ converges to the density of $\mathrm{N}(\Sigma^{-1}\mathbf{H},2\Sigma^{-1})$.

This verifies the crucial assumption 1.2 of Hannig (2008). Moreover, the equal-tailed region satisfies assumption 1.3 of Hannig (2008). The rest of the proof is identical to the proof of theorem 1 of Hannig (2008).

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