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CARMA(p,q) generalized random processes

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ABSTRACT

There is now a vast literature on the theory and applications of generalized random processes, pioneered by Itô (1953), Gel'fand (1955) and Yaglom (1957). In this note we make use of the theory of generalized random processes as defined in the book of Gel'fand and Vilenkin (1964) to extend the definition of continuous-time ARMA(p,q) processes to allow $q \geq p$, in which case the processes do not exist in the classical sense. The resulting CARMA generalized random processes provide a framework within which it is possible to study derivatives of CARMA processes of arbitrarily high order.

1. Introduction

A Gaussian continuous-time ARMA process with autoregressive order p and moving average order q (henceforth denoted CARMA(p,q)) can be defined formally, for $p > q \geq 0$, as a stationary solution of the stochastic differential equation

$$a(D)Y_t = b(D)DW_t, \quad t \in \mathbb{R}, \tag{1.1}$$

where D denotes differentiation with respect to t, $a(\cdot)$ and $b(\cdot)$ are the polynomials

$$a(z) = a_0 z^p + a_1 z^{p-1} + \cdots + a_p, \tag{1.2}$$

$$b(z) = b_0 + b_1 z + \cdots + b_q z^q, \tag{1.3}$$

where $a_0 := 1$ and $W := (W_t)_{t \in \mathbb{R}}$ is standard Brownian motion (i.e. W has continuous sample paths with time-homogeneous independent Gaussian increments, $W_0 = 0$ and W_1 is distributed as N(0,1)). Eq. (1.1) is the natural continuous-time analogue of the p th-order linear difference equations used to define a discrete-time ARMA process (see e.g. Brockwell and Davis, 1991). However, since the derivatives on the right-hand side of (1.1) do not exist as random functions, Eq. (1.1) is interpreted, when $q < p$, in the state-space form

$$Y_t = \mathbf{b}'\mathbf{X}_t, \quad t \in \mathbb{R}, \tag{1.4}$$

where $\mathbf{X} = (\mathbf{X}_t)_{t \in \mathbb{R}}$ is an $\mathbb{R}^p$-valued process satisfying the Itô equation

$$d\mathbf{X}_t = A\mathbf{X}_t\, dt + \mathbf{e}_p\, dW_t \tag{1.5}$$

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or equivalently

$$\mathbf{X}_t = e^{A(t-s)}\mathbf{X}_s + \int_s^t e^{A(t-u)}\mathbf{e}_p\, dW_u, \quad \forall s \le t \in \mathbb{R}, \tag{1.6}$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{e}_p = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix},$$

where $b_j := 0$ for $j > q$. For p=1 the matrix A is to be understood as $A = (-a_1)$.

In a recent paper of Brockwell and Lindner (2009, Theorem 4.2), it is shown that if all singularities of the meromorphic function $z \mapsto b(z)/a(z)$ on the imaginary axis are removable (i.e. if $a(\cdot)$ has a zero λ of multiplicity $\mu(\lambda)$ on the imaginary axis then $b(\cdot)$ has a zero at λ of multiplicity greater than or equal to $\mu(\lambda)$) then the strictly stationary solution of (1.4) and (1.5) is unique and given by

$$Y_t = \mathbf{b}'\mathbf{X}_t = \mathbf{b}' \int_{-\infty}^{\infty} \mathbf{g}(t-u)\, dW_u, \quad t \in \mathbb{R}, \tag{1.7}$$

where

$$\mathbf{g}(t) = \mathbf{l}(t)\mathbf{1}_{[0,\infty)}(t) - \mathbf{r}(t)\mathbf{1}_{(-\infty,0]}(t) \tag{1.8}$$

and $\mathbf{l}(t)$ and $\mathbf{r}(t)$ are the sums of the residues of the column vector $e^{zt}[1 \;\; z \;\; \cdots \;\; z^{p-1}]'/a(z)$ at the zeroes of $a(\cdot)$ with strictly negative and strictly positive real parts, respectively. Moreover there exist vectors $\mathbf{l}(0)$ and $\mathbf{r}(0)$ such that

$$\mathbf{l}(t) = e^{At}\mathbf{l}(0) \quad \text{and} \quad \mathbf{r}(t) = e^{At}\mathbf{r}(0). \tag{1.9}$$

(The results of Brockwell and Lindner apply also to the more general case in which W is replaced by a Lévy process L with $E\log^+|L_1| < \infty$.) We shall assume in this paper that $a(\cdot)$ has no zeroes on the imaginary axis. This is a necessary and sufficient condition for the existence of the CARMA(p,0) process corresponding to the autoregressive polynomial $a(z)$. It also implies (see Brockwell and Lindner, 2009) that

$$\mathbf{l}(t) + \mathbf{r}(t) = e^{At}\mathbf{e}_p, \quad t \in \mathbb{R}. \tag{1.10}$$

It is clear from (1.5) that the jth component, $j = 2, \ldots, p$, of the random vector $\mathbf{X}$ is the $(j-1)$st mean square derivative of the solution (1.7) with $\mathbf{b} = \mathbf{e}_1 = [1,0,\ldots,0]'$, i.e. of the CARMA($p$,0) process with autoregressive polynomial $a(z)$. This is the unique stationary solution $(X_t)_{t \in \mathbb{R}}$ of (1.4) and (1.5) with $\mathbf{b} = \mathbf{e}_1$. Thus, if $q < p$, the unique stationary solution of (1.4) and (1.5) is given by

$$Y_t = (b_0 + b_1 D + \cdots + b_q D^q) X_t, \quad t \in \mathbb{R}, \tag{1.11}$$

where D in (1.11) denotes mean square differentiation and both X and Y are stationary random processes in the usual sense. In other words, if $q < p$, the unique stationary solution of (1.4) and (1.5) is given by (1.11) with

$$X_t = \int_{-\infty}^{\infty} g_0(t-u)\, dW_u, \quad t \in \mathbb{R}, \tag{1.12}$$

where

$$g_0(t) = \mathbf{e}_1'[\mathbf{l}(t)\mathbf{1}_{[0,\infty)}(t) - \mathbf{r}(t)\mathbf{1}_{(-\infty,0]}(t)]. \tag{1.13}$$

The kernel g_0 can also be expressed (see Brockwell and Lindner, 2009) as

$$g_0(t) = \left(\sum_{\lambda:\Re\lambda < 0} \sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k e^{\lambda t} \mathbf{1}_{(0,\infty)}(t) - \sum_{\lambda:\Re\lambda > 0} \sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k e^{\lambda t} \mathbf{1}_{(-\infty,0)}(t) \right), \quad t \in \mathbb{R}, \tag{1.14}$$

where the sums are over the zeroes of $a(\cdot)$, $\mu(\lambda)$ is the multiplicity of the zero λ and the coefficients $c_{\lambda k}$ are determined by the expansion

$$\sum_{k=0}^{\mu(\lambda)-1} c_{\lambda k} t^k e^{\lambda t} = \frac{1}{(\mu(\lambda)-1)!}[D_z^{\mu(\lambda)-1}((z-\lambda)^{\mu(\lambda)} e^{zt}/a(z))]_{z=\lambda},$$

where D_z denotes differentiation with respect to z. (If $\mu(\lambda) = 1$ the last expression reduces to $e^{\lambda t}/a'(\lambda)$, where a' is the derivative of a.)

In order to extend the definition of CARMA(p,q) processes to include the possibility that $q \ge p > 0$, the representation of Y given in (1.11), (1.12) and (1.13) provides a natural starting point. In the following section we view X as defined in (1.12) as the convolution of the kernel g_0 with the derivative of W, regarded as a generalized random process (GRP) in the sense of Gel'fand and Vilenkin (1964, pp. 242–243), i.e. a continuous linear mapping from the space K of infinitely differentiable functions on $\mathbb{R}$ with compact support into the set of random variables on the probability space where W is defined. The convolution X is also a GRP so that the derivatives of X appearing in the (1.11) of Y, and hence Y itself, are well-defined GRPs reducing to random processes when $q<p$.

In Section 3 we consider the special case $p=1$, which clearly illustrates the explicit form of Y when $q \geq p$ and highlights the distinction between the cases $q<p$ and $q \geq p$. Section 4 deals with the connection with discrete-time ARMA(p,q) processes and Section 5 contains some concluding remarks.

2. CARMA(p,q) generalized random processes

There is now a vast literature on the theory and applications of generalized random processes, pioneered by Itô (1953), Gel'fand (1955) and Yaglom (1957). A generalized random process (GRP), in the terminology of Gel'fand and Vilenkin (1964), is a continuous linear functional V on K, where continuity in this context means that convergence of ϕ_{nj} to ϕ_j, $j=1,\dots,k$, in the topology of K (see Gel'fand and Vilenkin, p. 20) implies the corresponding convergence in distribution

$$(V(\phi_{n1}),V(\phi_{n2}),\dots,V(\phi_{nk})) \Rightarrow (V(\phi_1),V(\phi_2),\dots,V(\phi_k)). \tag{2.1}$$

If V is any GRP then its derivative is also a GRP, defined by

$$V^{(1)}(\phi)=V(-\phi'), \quad \phi \in K. \tag{2.2}$$

The GRP corresponding to standard Brownian motion is the continuous linear functional $W(\phi) := \int \phi(t)W(t)\,dt$, with corresponding GRP derivative,

$$W^{(1)}(\phi) := \int \phi(t)\,dW_t, \quad \phi \in K. \tag{2.3}$$

(All integrals, unless specified otherwise, are over $\mathbb{R}$.) The convolution of $W^{(1)}$ with the kernel g_0 is also a well-defined linear functional

$$(W^{(1)} * g_0)(\phi) := \int (\hat{g}_0 * \phi)(t)\,dW_t, \quad \phi \in K, \tag{2.4}$$

where g_0 was defined in (1.13), $\hat{g}_0(t) := g_0(-t)$ and $(\hat{g}_0 * \phi)(t) := \int \hat{g}_0(t-u)\phi(u)\,du$, $t \in \mathbb{R}$.

Since the convolution (2.4) is the GRP analogue of the representation (1.12) of the CARMA(p,0) process $(X_t)_{t\in\mathbb{R}}$, we define the CARMA(p,0) generalized random process X as

$$X(\phi)=(W^{(1)} * g_0)(\phi)=\int (\hat{g}_0 * \phi)(t)\,dW_t, \quad \phi \in K. \tag{2.5}$$

Changing the order of integration in (2.5) shows that we can also write

$$X(\phi)=\int X(u)\phi(u)\,du, \quad \phi \in K.$$

The derivative of any GRP V is also a GRP, defined by

$$V^{(1)}(\phi)=V(-\phi'), \quad \phi \in K. \tag{2.6}$$

Although the process $(X_t)_{t\in\mathbb{R}}$ has mean square derivatives of order only up to $p-1$, the corresponding GRP X has derivatives of all orders, each of which is a GRP. The jth of these will be denoted by $X^{(j)}$. Our next step is to evaluate the derivatives $X^{(j)}$ in order to express the GRP Y defined by (1.11) in terms of Brownian motion and its derivatives.

For this purpose it is convenient to introduce the vector-valued GRP $\mathbf{X}$, defined as the convolution (cf. (2.4)),

$$\mathbf{X}(\phi) := (W^{(1)} * \mathbf{g})(\phi)=\int (\hat{\mathbf{g}} * \phi)(t)\,dW_t, \quad \phi \in K, \tag{2.7}$$

where the p-vector $\mathbf{g}$ was defined in (1.8), $\hat{\mathbf{g}}(t) := \mathbf{g}(-t)$ and $(\hat{\mathbf{g}} * \phi)(t) := \int \hat{\mathbf{g}}(t-u)\phi(u)\,du$, $t \in \mathbb{R}$. Notice that $\mathbf{X}$ has GRP derivatives $\mathbf{X}^{(j)}$ of all orders and that the GRP derivatives of the CARMA(p,0) process defined by (2.5) are related to those of $\mathbf{X}$ by the relations

$$X^{(j)}=\mathbf{e}_1'\mathbf{X}^{(j)}, \quad j=0,1,2,\dots, \tag{2.8}$$

where $\mathbf{e}_1$ is the p-vector, $\mathbf{e}_1=[1\ \ 0\ \ \cdots\ \ 0]'$. The following proposition shows that the GRP $\mathbf{X}$ as defined in (2.7) satisfies Eq. (1.5), interpreted as a relation between generalized random processes.

Proposition 2.1.

$$\mathbf{X}^{(1)}=A\mathbf{X}+W^{(1)}\mathbf{e}_p=A(W^{(1)} * \mathbf{g})+W^{(1)}\mathbf{e}_p. \tag{2.9}$$

Proof. The GRP derivative of $\mathbf{X}$ is

$$\mathbf{X}^{(1)}(\phi)=-\mathbf{X}(\phi')=-(W^{(1)} * \mathbf{g})(\phi')=-\int (\hat{\mathbf{g}} * \phi')(t)\,dW_t, \quad \phi \in K,$$

where $(\hat{\mathbf{g}}*\phi')(t)=\int\hat{\mathbf{g}}(t-u)\phi'(u)\,du$. From (1.8) and (1.9) we can write

$$\mathbf{g}(t)=e^{At}(\mathbf{l}(0)\mathbf{1}_{[0,\infty)}(t)-\mathbf{r}(0)\mathbf{1}_{(-\infty,0]}(t)),$$

and hence, substituting in the definition of $\hat{\mathbf{g}}*\phi'$ and integrating by parts,

$$\begin{aligned}(\hat{\mathbf{g}}*\phi')(t)&=\int_t^\infty e^{-A(t-u)}\mathbf{l}(0)\,d\phi(u)-\int_{-\infty}^t e^{-A(t-u)}\mathbf{r}(0)\,d\phi(u)\\&=-A(\hat{\mathbf{g}}*\phi)(t)-\phi(t)(\mathbf{l}(0)+\mathbf{r}(0)).\end{aligned}$$

Integrating both sides with respect to dW_t and using (1.10), we obtain

$$\mathbf{X}^{(1)}(\phi)=A\int(\hat{\mathbf{g}}*\phi')(t)\,dW_t+W^{(1)}(\phi)\mathbf{e}_p=A(W^{(1)}*\mathbf{g})(\phi)+W^{(1)}(\phi)\mathbf{e}_p,$$

which, since $\mathbf{X}=W^{(1)}*\mathbf{g}$, is equivalent to the statement of the proposition. □

Corollary 2.2.

$$\mathbf{X}^{(j)}=A^j\mathbf{X}+\sum_{k=0}^{j-1}W^{(j-k)}A^k\mathbf{e}_p,\quad j=0,1,2,\ldots. \tag{2.10}$$

Proof. For $j=1$ this is a restatement of the proposition. To establish the general result by induction, assume that it holds for $j=m$. Differentiating each side of the equation with $j=m$ and using Proposition 2.1 immediately establishes the validity for $j=m+1$. □

Corollary 2.3.

$$X^{(j)}=\mathbf{e}_1'A^j\mathbf{X}+\sum_{k=p-1}^{j-1}W^{(j-k)}\mathbf{e}_1'A^k\mathbf{e}_p,\quad j=0,1,2,\ldots, \tag{2.11}$$

where $\mathbf{X}=W^{(1)}*\mathbf{g}$ *and* $\mathbf{g}$ *was defined in* (1.8).

Proof. The result follows straight from (2.8), (2.10) and the fact that $\mathbf{e}_1'A^k\mathbf{e}_p=0$ for $k<p-1$. The sum in (2.11) is zero for $j<p-1$ and $W^{(1)}$ for $j=p$. □

We are now ready to define the CARMA(p,q) generalized random process with autoregressive and moving average polynomials $a(z)$ and $b(z)$, respectively, and without the constraint that $q<p$. We continue to assume, however, that $a(\cdot)$ is non-zero on the imaginary axis.

Definition 2.4. The CARMA(p,q) generalized random process Y with autoregressive polynomials $a(z)$ and $b(z)$ as in (1.2) and (1.3) is defined as

$$Y=\begin{cases}\sum_{j=0}^{q}b_jX^{(j)} & \text{if } p>0,\\ \sum_{j=0}^{q}b_jW^{(j+1)} & \text{if } p=0,\end{cases} \tag{2.12}$$

with $X^{(j)}$ specified in (2.11).

Proposition 2.5. *The CARMA*(p,q) *generalized random process Y defined in* (2.12) *satisfies Eq.* (1.1) *with the operator D interpreted as differentiation in the GRP sense.*

Proof. If $p>0$ we have

$$\begin{aligned}\sum_{k=0}^{p}a_kY^{(p-k)}&=\sum_{k=0}^{p}a_k\sum_{j=0}^{q}b_jX^{(p-k+j)}\\&=\sum_{j=0}^{q}b_j\sum_{k=0}^{p}a_kX^{(p-k+j)}=\sum_{j=0}^{q}b_jW^{(j+1)},\end{aligned}$$

since, by the last component equation of (2.9), $\sum_{k=0}^{p}a_kX^{(p-k)}=W^{(1)}$. In the case $p=0$, (2.12) is just a restatement of (1.1). □

Remark. The CARMA(p,q) GRP as defined in (2.12) is strictly stationary in the sense that if S_u denotes the shift operator on K, i.e. if $S_u\phi(\cdot)=\phi(\cdot+u)$, then $(Y(\phi_1),\ldots,Y(\phi_k))$ has the same distribution as $(Y(S_u\phi_1),\ldots,Y(S_u\phi_k))$ for all positive integers k, for all $\phi_1,\ldots,\phi_k\in K$ and for all $u\in\mathbb{R}$. This follows from independence and homogeneity of the increments of standard Brownian motion and the preservation of strict stationarity under differentiation and convolution.

3. The case p=1

The general case is perhaps most clearly illustrated in the case p=1, when the autoregressive polynomial is $a(z)=z-\lambda$ with $\lambda \in \mathbb{R}\backslash\{0\}$. If q=0 then Y is the stationary Ornstein–Uhlenbeck process. In general we have, from (2.11),

$$X^{(j)}=\lambda^j X+\sum_{k=0}^{j-1}\lambda^k W^{(j-k)}, \quad j=0,1,2,\ldots, \tag{3.1}$$

where

$$\begin{aligned} X(\phi) &= \iint \hat{g}(t-u)\phi(u)\,du\,dW_t \\ &= \int X(u)\phi(u)\,du, \end{aligned}$$

with $X(t)=\int g(t-u)\,dW_u$, $\hat{g}(t)=g(-t)$ and $g(t)=e^{\lambda t}(\mathbf{1}_{[0,\infty)}(t)l(0)+\mathbf{1}_{(-\infty,0]}(t)r(0))$, where $l(0)=\mathbf{1}_{(-\infty,0)}(\lambda)$ and $r(0)=\mathbf{1}_{(0,\infty)}(\lambda)$.

Hence, if the moving average polynomial is $b(z)=b_0+b_1z+\cdots+b_qz^q$, then from (2.12) we see that the CARMA(1,q) generalized random process Y is given by

$$Y=b(\lambda)X+\sum_{j=1}^{q}\sum_{k=j}^{q}b_k\lambda^{k-j}W^{(j)}.$$

This expression contains a "regular" component $b(\lambda)X$, interpretable as a random function, and derivatives of W up to order $q+1-p$. A similar structure is found also in the general case.

Example 3.1. The velocity of a particle whose position at time t is specified by a stationary Ornstein–Uhlenbeck process with $a(z)=z-\lambda$ and $b(z)=b_0$ does not exist in the classical random process sense; however, it does exist as a CARMA(1,1) GRP, denoted by Y, with $a(z)=z-\lambda$ and $b(z)=b_0z$. Although the velocity at time t does not exist, it makes perfectly good sense to observe the random variable, $Y(\phi)$ for some function $\phi \in K$. In particular, if ϕ is chosen to be an approximation in K to the function $\mathbf{1}_{(t,t+\Delta)}(\cdot)/\Delta$, $Y(\phi)$ will be an approximation to the average velocity over the time interval $(t,t+\Delta)$. Higher derivatives of the Ornstein–Uhlenbeck process and of more general CARMA processes may be treated analogously.

4. The relation with discrete-time ARMA processes

Define $\phi(z) := a(\delta^{-1}(1-z))$ and $\theta(z) := b(\delta^{-1}(1-z))$ where $a(z)$ and $b(z)$ are the polynomials defined in (1.2) and (1.3). Denote by Λ the set of distinct zeroes of the polynomial $a(z)$ and by $\mu(\lambda)$ the multiplicity of the zero λ. It will be assumed, as in previous sections, that none of the zeroes λ lies on the imaginary axis. We shall assume also that $0<\delta<c$ where c is small enough to ensure that none of the zeroes $1-\delta\lambda$ of the polynomial $\phi(z)$ lies on the unit circle. Let $(Y_n)_{n\in\mathbb{Z}}$ be the unique stationary solution of the ARMA equations

$$\phi(B)Y_n=\theta(B)\delta^{-1/2}Z_n, \quad n\in\mathbb{Z}, \tag{4.1}$$

where B is the backward shift operator and $(Z_n)_{n\in\mathbb{Z}}$ is i.i.d. Gaussian with mean 0 and variance 1. Then

$$Y_n=\theta_0X_n+\theta_1X_{n-1}+\cdots+\theta_qX_{n-q}, \quad n\in\mathbb{Z}, \tag{4.2}$$

where X_n is the AR(p) process defined by Eq. (4.1) with $\theta(B)=1$.

Under the assumptions made in the previous paragraph $\phi(z)^{-1}$ has the Laurent expansion, $\phi(z)^{-1}=a(\delta^{-1}(1-z))^{-1}=\sum_{k=-\infty}^{\infty}g_k^\delta z^k$, valid in an annulus containing the unit circle, where

$$g_k^\delta=\left(\sum_{\lambda:|1-\lambda\delta|>1}p_\lambda^\delta(k)\mathbf{1}_{[0,\infty)}(k)-\sum_{\lambda:|1-\lambda\delta|<1}p_\lambda^\delta(k)\mathbf{1}_{(-\infty,0)}(k)\right), \quad k\in\mathbb{Z},$$

the sums are over the distinct zeroes λ of $a(z)$ and

$$p_\lambda^\delta(k)=-\frac{(-\delta)^p}{(\mu(\lambda)-1)!}\left[D_z^{\mu(\lambda)-1}\left((z-1+\delta\lambda)^{\mu(\lambda)}z^{-k-1}\Big/\prod_\lambda(z-1+\delta\lambda)^{\mu(\lambda)}\right)\right]_{z=1-\delta\lambda},$$

where D_z denotes differentiation with respect to z.

Let us now place the processes on the lattice $\{n\delta, n\in\mathbb{Z}\}$. Define $X_t^\delta = X_{[t/\delta]}$, $Y_t^\delta = Y_{[t/\delta]}$, and $g^\delta(t)=\delta^{-1}g^\delta_{[t/\delta]}$, with $[x]$ being the integer part of x. Since $X_k=\delta^{-1/2}\sum_{l=-\infty}^{\infty} g^\delta_{k-l}Z_k$, we have for a suitably chosen standard Brownian motion

$$X_n = X^\delta_{n\delta} \stackrel{\mathcal{D}}{=} \int_{-\infty}^{\infty} g^\delta(n\delta - s)\,dW_s, \tag{4.3}$$

where the superscript $\mathcal{D}$ means equality in distribution of the sequence $\{X^\delta_{n\delta}, n=1,2,\ldots\}$ and the sequence on the right.

After a careful calculation we see that, as $\delta\to 0$,

$$\delta^{-1}p_\lambda^\delta([t/\delta]) \to \frac{1}{(\mu(\lambda)-1)!}[D_z^{\mu(\lambda)-1}((z-\lambda)^{\mu(\lambda)}e^{zt}/a(z))]_{z=\lambda} \tag{4.4}$$

and consequently $g^\delta(t)\to g_0(t)$ with g_0 as in (1.14).

Consider now X^δ as a generalized process, i.e. $X^\delta(\phi)=(W^{(1)} * g^\delta)(\phi)$. Eqs. (2.5), (4.3) and (4.4) imply that X^δ converges to X in the sense of finite dimensional distributions, i.e. for any fixed $\phi_1,\ldots,\phi_k \in K$, $(X^\delta(\phi_1),\ldots,X^\delta(\phi_k))$ converges in distribution to $(X(\phi_1),\ldots,X(\phi_k))$ as $\delta\to 0$.

From (4.2),

$$Y^\delta(t) = \sum_{i=0}^{q} b_i \delta^{-i} \sum_{j=0}^{i} \binom{i}{j} (-1)^j X^\delta_{t-j\delta}.$$

If we again interpret $Y^\delta(t)$ as a generalized process we get

$$Y^\delta(\phi) = \sum_{i=0}^{q} b_i\delta^{-i}\sum_{j=0}^{i}\binom{i}{j}(-1)^j X^\delta(\phi(\cdot+j\delta)) = \sum_{i=0}^{q} b_i X^\delta\left(\delta^{-i}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\phi(\cdot+j\delta)\right).$$

However, $\delta^{-i}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\phi(t+j\delta)\to(-1)^i\phi^{(i)}(t)$ and (2.12) implies Y^δ converges to $\sum_{i=0}^{q} b_i X((-1)^i\phi^{(i)})=Y$ in the sense of finite dimensional distributions.

This establishes the sense in which Y^δ, for small δ, approximates the CARMA generalized random process Y defined in Section 2.

5. Conclusions

Yaglom (1957) considered classes of generalized random fields in his study of turbulence and Gel'fand and Vilenkin (1964) gave a general account of generalized random fields and generalized random processes. In this note we have used the concept of generalized random process, together with a recent characterization of the unique stationary solution of the CARMA equations, to extend the definition of Gaussian CARMA(p,q) processes to include the possibility that $q\geq p$. We remark finally that for $q<p$ the CARMA(p,q) process Y can also be regarded as the GRP defined by $Y(\phi) := \int \phi(t)Y(t)\,dt$, $\phi\in K$, which, with Definition 2.4, provides a unified framework for the study of CARMA(p,q) processes and their derivatives regardless of whether or not $p>q$.

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