STOR 655 – May 5, 2016 Name:________________________

Final Exam

All problem parts have equal weight. In budgeting your time expect that some problems will take longer than others. You can use results of earlier parts in later parts.
Remember, answers without proper justification will not receive full credit!

1. Assume $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \dots \begin{pmatrix} X_n \\ Y_n \end{pmatrix}$ are i.i.d. bivariate normal$\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \rho & \rho \\ \rho & 1 \end{pmatrix} \right)$, $0 < \rho < 1$.

   (a) Prove that the Fisher information $I(\theta) = \frac{1}{2\rho^2} + \frac{1}{2(1-\rho)^2}$.

   (b) Consider $R_n = \frac{1}{n} \sum_{i=1}^n X_i Y_i$ and $V_n = \frac{1}{n} \sum_{i=1}^n X_i^2$. Are they consistent estimators of ρ?

   (c) What is the asymptotic distribution of these estimators? Are they asymptotically normal?

   (d) Is $\frac{R_n^3}{V_n^2}$ a strongly consistent estimator of ρ? What is its asymptotic distribution?

   (e) Is any of the three estimators above asymptotically efficient? If not, improve at least one of them by scoring.

2. Let $X_1, \ldots, X_n$ i.i.d. Uniform$(0, \theta)$, $\theta > 0$.

   (a) Find $\hat{\theta}_n$ the MLE. Is it a consistent estimator of θ?

   (b) What is the asymptotic distribution of $\hat{\theta}_n$? is it asymptotically normal? (Hint: Consider convergence of $n(\theta_0 - \hat{\theta}_n)$.)

   (c) Find the Bayes posterior distribution using an improper prior $\pi(\theta) = 1/\theta$.

   (d) Consider rescaling the posterior distribution by the change of variable $\zeta = n(\theta - \hat{\theta}_n)$. What does the rescaled posterior distribution converges to as $n \to \infty$? Is this convergence in L_1?

   (e) Consider parametric bootstrap estimator of the $T_n = \frac{\theta_0 - \hat{\theta}_n}{\hat{\theta}_n}$ by $T_n^\star = \frac{\hat{\theta}_n - \hat{\theta}_n^\star}{\hat{\theta}_n^\star}$, where $\hat{\theta}_n^\star$ is based on bootstrap samples of $X_1^\star, \ldots, X_n^\star$ i.i.d. Uniform$(0, \hat{\theta}_n)$. Does it lead to correct inference?
   Would the answer change if we used non-parametric bootstrap, i.e., the bootstrap samples to be resamples of the original data?

3. Let $Z \sim N(0, I)$ be standard multivariate normal random variable. Show that $Z^\top PZ$ has χ^2_r distribution if and only if P is a projection matrix of rank r.