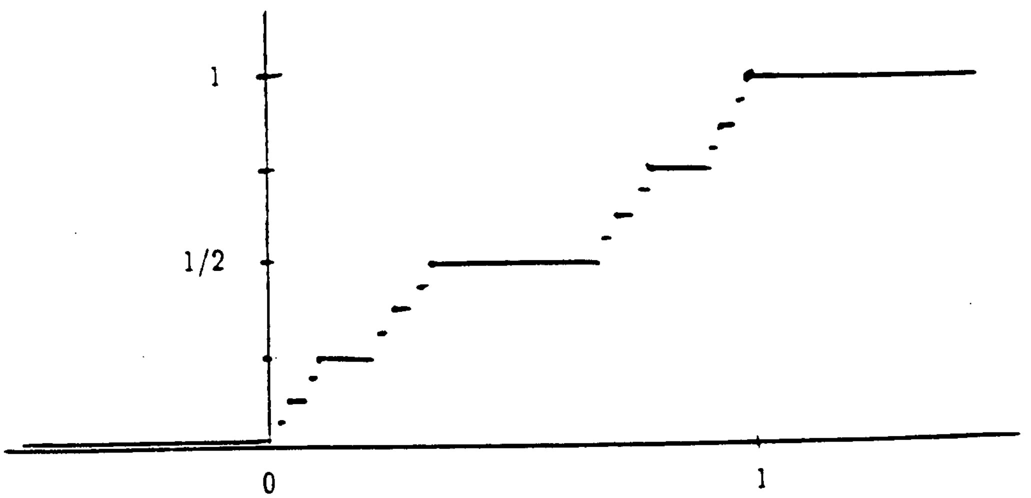
EXAMPLE 14 Define cdf $F(\cdot)$ as follows: For $x < 0$, define $F(x) = 0$ and for $x \geq 1$, define $F(x) = 1$. For $\frac{1}{3} \leq x < \frac{2}{3}$ define $F(x) = \frac{1}{2}$; for $\frac{1}{9} \leq x < \frac{2}{9}$, define $F(x) = \frac{1}{4}$ and for $\frac{7}{9} \leq x < \frac{8}{9}$ define $F(x) = \frac{3}{4}$; for $\frac{1}{27} \leq x < \frac{2}{27}$, define $F(x) = \frac{1}{8}$, for $\frac{7}{27} \leq x < \frac{8}{27}$, define $F(x) = \frac{3}{8}$, for $\frac{19}{27} \leq x < \frac{20}{27}$ define $F(x) = \frac{5}{8}$, and for $\frac{25}{27} \leq x < \frac{26}{27}$ define $F(x) = \frac{7}{8}$; etc. Always define $F(x)$ on the middle thirds of the intervals where $F(x)$ has yet to be defined and give $F(x)$ the value half way between the preceding and succeeding defined values. The sum of these middle third intervals over which $F(x)$ is so defined, and flat, is given by $\frac{1}{3} + 2\left(\frac{1}{3}\right)^2 + 4\left(\frac{1}{3}\right)^3 + \cdots = \frac{1}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right) = \frac{1}{3}\frac{1}{1-\frac{2}{3}} = 1$. So $F(\cdot)$ has been defined to be flat over intervals of length one within the $[0, 1)$ interval. Complete the definition of $F(\cdot)$ by invoking the requisite right continuity. A cdf has been defined; it can be argued that it has no jumps so is in fact continuous. Its derivative is zero for all x for which $F(x)$ is flat, which is almost all x, so $F(x)$ cannot be written as the integral of its derivative, so $F(\cdot)$ is not absolutely continuous. It is appropriately called *singular* continuous. In mathematics, this $F(x)$ is known as the *Cantor function*, so we call it the *Cantor* cdf. A sketch is attempted below.



Definition 10 Singular continuous cumulative distribution function A cdf $F(\cdot)$ is called *singular continuous* if $F(x)$ is continuous but $\frac{dF(x)}{dx} = 0$ for almost all x. ////

Note that property (iv) of cdfs, namely, the general decomposition of an arbitrary cdf is now meaningful since all items in the decomposition have been defined.