STOR 654 – October 18, 2016   Name:_____

All problem parts have equal weight. In budgeting your time expect that
some problems will take longer than others. Remember, answers without
proper justification will not receive full credit!

1. Prove that if $EX^2 < \infty$ then $P(X - EX \geq t) \leq \frac{\text{Var } X}{\text{Var } X + t^2}$ for all $t > 0$.
   (Hint: $t \leq E\{[(t - (X - EX)]I_{\{X-EX<t\}}\}$ might be useful).

2. Let $X_1, X_2, \ldots, X_n$ be i.i.d. $\text{Beta}(a, 2a)$. where $a > 0$ is an unknown parameter.

   (a) Is this exponential family?

   (b) Find a minimal sufficient statistic T. Is it complete?

   (c) Find the MM estimator of a.

   (d) Find the MLE of a. (Hint: There might not be a closed form solution. If that is the case describe what equation would you solve numerically.)

   (e) Find the conditional expectation $E[\sum_{i=1}^n X_i \mid T]$.

3. Let $X_1, \ldots, X_n$ be independent $X_i \sim \text{Bernoulli}(i/(n+1))$. Denote the sample mean $\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i$.

   (a) Find $E\bar{X}_n$ and $\text{Var}\,\bar{X}_n$.

   (b) Prove or disprove $P(\bar{X}_n \leq 1/4) \leq e^{-\frac{n}{8}}$