STOR 435.001 Lecture 16

Properties of Expectation - I

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Motivation

- Recall we found joint distributions to be pretty complicated objects. Need various tools from combinatorics calculus etc to understand these.
- A useful summary of random variables is *expectation*.
- Also found this to be super useful for understanding moment generating functions.
- This lecture: more properties of expectations, especially as it relates to more than one random variable.
- *In class: can show you some fun examples. Do the HW problems to really get a feeling for the material.*

Key concepts in these slides

1. Basic definition: *Slide 4*.
2. Consequences: *Slide 5*.
3. Expected *number of events* that occur: *Slides 9 and 12*.
4. Variance and covariance: definition and properties. *Slides 14, 15, 16*.
5. Correlation: *Slide 20*.

Properties of expectation that we have already seen

Reminder

Let $g(x,y)$ be a function. If X, Y are discrete with p.m.f. $p(x,y)$, then

$$Eg(X,Y) = \sum_x \sum_y g(x,y)p(x,y).$$

If X, Y are jointly continuous with density $f(x,y)$, then

$$Eg(X,Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy.$$

Small note: when showing you why some of the properties below are true, I will (for simplicity) use continuous random variables. The same "proofs" go through by replacing integration with "summation" for discrete random variables.

Consequence 1

$E(X+Y) = EX + EY$ and more generally $E\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} EX_i$.

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Consequence 2

If X and Y are **independent** and $h_1(x), h_2(y)$ are two functions, then $Eh_1(X)h_2(Y) = Eh_1(X)Eh_2(Y)$.

We verified this when we used moment generating functions.

Properties of expectation

A random walk in the plane: Suppose a particle is initially located at the origin in the plane. Now suppose that it undergoes a sequence of steps of fixed length, but in a completely random direction. One way to model this is as follows: suppose that the new position $S_n \in \mathbb{R}^2$ after each step n is a unit distance from the previous position and at an angle of orientation from the previous position that is uniformly distributed over $(0, 2\pi)$. Thus

$$S_n = S_{n-1} + (X_n, Y_n),$$

where $X_n = \cos(\theta_n)$ and $Y_n = \sin(\theta_n)$ and θ_n is uniform $[0, 2\pi]$. Calculate the expected **square of the distance** from the origin after n steps.

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Properties of expectation

Example: Let X be a binomial random variable with parameters n and p. Compute EX.

Expected number of events that occur: It is a common situation where we want to compute EX with X being the *number* of something. Moreover, it is often the case that for some events $A_1, A_2, \dots, A_n$, X is the number of these events that occur (e.g. $A_i = \{\text{success on trial } i\}$). Then,

$$X = \sum_{i=1}^{n} I_{A_i} \quad \text{with} \quad I_{A_i} = \begin{cases} 1, & \text{if } A_i \text{ occurs,} \\ 0, & \text{if not} \end{cases}$$

and

$$EX = \sum_{i=1}^{n} EI_{A_i} = \sum_{i=1}^{n} (1 \cdot P(A_i) + 0 \cdot P(A_i^c)) = \sum_{i=1}^{n} P(A_i).$$

Properties of expectation

Example: Ten hunters are waiting for ducks to fly by. When a flock of ducks flies overhead, the hunters shoot at the same time, but each chooses his target at random, independently of the others. If each hunter independently hits his target with probability p, compute the expected number of ducks that escape unhurt when a flock of size 20 flies overhead.

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Properties of expectation

Example: N people arrive separately to a professional dinner. Upon arrival, each person looks to see if he or she has any friends among those present. That person then sits either at the table of a friend or at an unoccupied table if none of those present is a friend. Assuming that each of the $\binom{N}{2}$ pairs of people is, independently, a pair of friends with probability p, find the expected number of occupied tables.

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Higher-order moments of number of events that occur: For some events $A_1, A_2, \ldots, A_n$, let

$$X = \sum_{i=1}^{n} I_{A_i} \quad \text{with} \quad I_{A_i} = \begin{cases} 1, & \text{if } A_i \text{ occurs,} \\ 0, & \text{if not} \end{cases}$$

be the number of these events that occur. Note that

$$\frac{X(X-1)}{2} = \binom{X}{2} = \sum_{i_1 < i_2} I_{A_{i_1}} I_{A_{i_2}}$$

is the number of pairs of events $A_1, A_2, \ldots, A_n$ where both events occur.

Higher-order moments of number of events that occur: More generally,

$$\binom{X}{k} = \sum_{i_1 < i_2 < \ldots < i_k} I_{A_{i_1}} I_{A_{i_2}} \ldots I_{A_{i_k}}$$

is the number of distinct subsets of k events that all occur. Hence,

$$E\binom{X}{k} = \sum_{i_1 < i_2 < \ldots < i_k} P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}).$$

Variances, covariances, correlations ⚠

Covariance between X and Y

$$Cov(X,Y) = E(X - EX)(Y - EY) = EXY - EXEY$$

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Motivation

- Covariance gives idea of the relationship between X and Y. Roughly speaking: covariance signifies tendency of two random variables to go up or down together relative to their expected value.
- Positive indicates that when X goes up, Y "tends" to go up. Example: pick a person at random. X is height, Y is weight.
- Negative: X goes up, Y "tends" to go down. Note everything is "random" so one can only talk about tendency, cannot guarantee that this will happen.

For spurious correlations see
`http://tylervigen.com/spurious-correlations`

Properties of expectation

Some basic properties: $Cov(X,Y) = Cov(Y,X)$, $Cov(X,X) = Var(X)$, $Cov(aX+b,Y) = aCov(X,Y)$.

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Important property

$$Cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(X_i, Y_j)$$

Consequence: $Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$

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Note 1: If X_i's are pairwise independent, then $Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$.

Note 2: If $X_i = 1$ if event A_i occurs and $= 0$ otherwise, then $Var(\sum_{i=1}^{n} X_i) = 2\sum_{i<j} P(A_i \cap A_j) + \sum_i P(A_i) - (\sum_i P(A_i))^2$.

Properties of expectation

Example: Let X be a binomial random variable with parameters n and p. Compute $Var(X)$.

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Example: Toss a fair coin 3 times. Let X be the number of heads, and Y be the number of tails. Find $Cov(X, Y)$.

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Properties of expectation

Problem: A group of 20 people consisting of 10 men and 10 women is randomly arranged into 10 pairs of 2 each. Compute the expectation and variance of the number of pairs that consist of a man and a woman.

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Correlation between X and Y

$$\rho(X,Y) = Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

Fact 1: $-1 \leq \rho(X,Y) \leq 1$

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Terminology: X and Y are called uncorrelated when $\rho(X,Y) = 0$.

Properties of expectation

Fact 2: For example, if $\rho(X, Y) = -1$, then $Y = -aX + b$ with $a > 0$.

☼ (Cauchy-Schwartz inequality)

Fact 3: $\rho(aX + b, Y) = \rho(X, Y)$ ($a > 0$)

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Note: $\rho(X, Y)$ measures the strength and direction of a linear relationship between X and Y (close to 1: strong linear, positive slope; close to -1: strong linear, negative slope).

Problem: Suppose we have a bivariate normal distribution

$$\begin{pmatrix} X \\ Y \end{pmatrix} = N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix} \right).$$

Calculate Cov(X, Y). Calculate $\rho(X, Y)$.

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The joint density

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left((\frac{x-\mu_X}{\sigma_X})^2 + (\frac{y-\mu_Y}{\sigma_Y})^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} \right)}$$

For a simpler way recall: To simulate

$$\begin{pmatrix} X \\ Y \end{pmatrix} = N\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}\right).$$

1. Step 1: Simulate independent Z_1, Z_2 standard Normal random variables.
2. Step 2: $X = \mu_X + \sigma_X Z_1$.
3. Step 3:

$$Y = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X) + Z_2\sigma_Y\sqrt{(1-\rho^2)}$$

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