STOR 435.001 Lecture 13

Jointly distributed Random Variables - I

Jan Hannig

UNC Chapel Hill

Jointly distributed random variables

Collections of 2 or more random variables $X_1, X_2, \ldots, X_n$. Interested in modeling relationships between them as well.

Examples:
$X_1 =$ price of Netflix Stock, $X_2 =$ price of Amazon stock, $X_3 =$ price of Google stock.
$X_1 =$ price of oil today, $X_2 =$ price of oil tomorrow, etc.
$X_1 =$ expenditures on food, $X_2 =$ expenditures on housing, etc.
$X_1 =$ cholesterol level, $X_2 =$ blood pressure, etc.
$X_1 =$ rainfall in NC, $X_2 =$ rainfall in VA, etc.

Most advanced statistical topics (time series analysis, multivariate analysis, multiple linear regression, factor models, etc) and probability topics (Markov chains, stochastic processes, etc) involve collections of random variables.

Jointly distributed random variables

Focus on: Two random variables X, Y. All probability questions about X and Y can be answered in terms of their joint c.d.f.

Joint cumulative distribution function (c.d.f.): $F(a, b) = P(X \leq a, Y \leq b)$, $-\infty < a, b < \infty$.

For example: F carries info about X, Y individually: e.g.

$F_X(a) =$

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But also: e.g.

$P(X > a, Y > b) =$

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$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =$

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Two broad classes of random variables:

1. Both X and Y are discrete: characterized through *joint probability mass function* (p.m.f.)
$$p(x, y) = P(X = x, Y = y).$$
2. X and Y are jointly continuous: there is a non-negative function $f(x, y)$, called *joint probability density function* (p.d.f.), such that, for any set C in the two-dimensional plane,
$$P((X, Y) \in C) = \int\int_{(x,y)\in C} f(x, y)dxdy.$$

Joint distributions of discrete random variables

Discrete random variable

1. Characterized by their joint probability mass function

$$p_{X,Y}(x, y) = P(X = x, Y = y).$$

2. If we are given the joint pmf then very easy to get the pmf of any one of the random variables. For example

$$p_X(x) = P(X = x) = P(X = x, \text{Y takes any value }) = \sum_y p_{X,Y}(x, y)$$

Sometimes referred to as the **marginal distribution of** X. The **same** as the distribution of X.

Expectations of functions of discrete random variables

If X, Y have joint pmf $p_{X,Y}$ and $g(x,y)$ is a function of the two variables (e.g $g(x,y) = x + y$ or $g(x,y) = \cos(x) + \sin(y)$) then

$$E(g(X,Y)) = \sum_{x,y} g(x,y) p_{X,Y}(x,y).$$

Special case

Suppose $g(x,y) = x$. Then we get

$$\mathbb{E}(X) = \sum_{x,y} x p_{X,Y}(x,y) = \sum_{x} x \left[\sum_{y} p_{X,Y}(x,y) \right] = \sum_{x} x p_X(x).$$

Thus to calculate the expected value of X we could either first calculate marginal pmf p_X of X and then calculate the expected value as before $E(X) = \sum_x x p_X(x)$ or directly calculate it using the joint pmf as above. Both will give us the **same** answer.

Problem 1: Two fair dice six faced dice are rolled. Find the joint probability mass function of X and Y when X is the largest value obtained on any die and Y is the sum of the values.

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Next: A number of notes for the jointly continuous case.
Note 1: $\int\int_{(x,y)\in C} f(x,y)dxdy$ is the volume under the surface $f(x,y)$ above the region C. In particular, when $f \equiv 1$,

$$\int\int_{(x,y)\in C} dxdy = \text{Area}(C).$$

Note 2: With $C = A \times B = \{(x,y) : x \in A, y \in B\}$,

$$P(X \in A, Y \in B) = \int_A dx \int_B dy\, f(x,y)$$

Note 3:

$$F(a,b) = \int_{-\infty}^{a} dx \int_{-\infty}^{b} dy\, f(x,y), \quad \frac{\partial^2}{\partial a \partial b} F(a,b) = f(a,b)$$

Note 4:

$$P(a < X \leq a + da, b < Y \leq b + db)$$
$$= \int_a^{a+da} dx \int_b^{b+db} dy\, f(x,y) \approx f(a,b) da db$$

for small da, db, if f is continuous at (a,b). Thus, $f(a,b)$ is a measure of how likely X, Y is near a, b.

Note 5: Each individual random variable is continuous. E.g.

$$P(X \in A) = P(X \in A, Y \in (-\infty, \infty)) = \int_A dx \int_{-\infty}^{\infty} dy\, f(x,y)$$

and hence the (*marginal*) density of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y) dy.$$

Similarly, $f_Y(y) = \int_{-\infty}^{\infty} f(x,y) dx$.

Expectations of functions of continuous random variables

If X, Y have joint pdf $f_{X,Y}$ and $g(x, y)$ is a function of the two variables (e.g $g(x, y) = x + y$ or $g(x, y) = \cos(x) + \sin(y)$) then

$$E(g(X, Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dy dx.$$

Special case

Suppose $g(x, y) = x$. Then we get

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y}(x, y) dy dx. = \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{X,Y}(x, y) dy \right] dx = \int_{-\infty}^{\infty} x f_X(x) dx.$$

Thus to calculate the expected value of X we could either first calculate marginal pdf f_X of X and then calculate the expected value as before $E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$ or directly calculate it using the joint pmf as above. Both will give us the **same** answer.

Example 1c(b): The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y}, & 0 < x < \infty, 0 < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

Compute $P(X < Y)$.

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Jointly distributed random variables

Example

Consider a circle of radius R, and suppose that a point within the circle is randomly chosen in such a manner that all regions within the circle of equal area are equally likely to contain the point. (In other words, the point is uniformly distributed within the circle.) If we let the center of the circle denote the origin and define X and Y to be the coordinates of the point chosen then, since (X, Y) is equally likely to be near each point in the circle, it follows that the joint density function of X and Y is given by

$$f(x,y) = \begin{cases} c & \text{if } x^2 + y^2 \leq R^2 \\ 0 & \text{if } x^2 + y^2 > R^2 \end{cases}$$

for some value of c.

- a Determine c.
- b Find the marginal density functions of X and Y.
- c Compute the probability that D, the distance from the origin of the point selected, is less than or equal to a.
- d Find $E[D]$.

Example cont'ed:

More than two random variables: The notions above can be extended to more than two random variables $X_1, X_2, \ldots, X_n$. For example, the joint c.d.f. is defined as

$$F(a_1, a_2, \ldots, a_n) = P(X_1 \leq a_1, X_2 \leq a_2, \ldots, X_n \leq a_n).$$

For discrete random variables we can talk about **joint pmf**

$$p(x_1, x_2, \ldots x_n) = P(X_1 = x_1, X_2 = x_2, \cdots X_n = x_n).$$

In the continuous case, the random variables $X_1, X_2, \ldots, X_n$ are jointly continuous if there is a non-negative function $f(x_1, x_2, \ldots, x_n)$, called joint probability density function (p.d.f.), such that, for any set C in the n-dimensional space,

$$P((X_1, X_2, \ldots, X_n) \in C) = \int\int_{(x_1, x_2, \ldots, x_n) \in C} \ldots \int f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots x_n.$$

Jointly distributed random variables

Definitions of marginal distributions, expectations of functions etc all similar to the 2 variable case.

Discrete case

So for example if $(X_1, \ldots X_n)$ are discrete with joint pmf p then to get the marginal pmf of X_2 we would get this from the pmf by summing over all other co-ordinates namely

$$p_{X_2}(x_2) = \sum_{x_1} \sum_{x_3} \cdots \sum_{x_n} p(x_1, x_2, x_3, \ldots, x_n)$$

Continuous case

If $(X_1, \ldots X_n)$ are continuous with joint pdf f then to get the marginal pdf of X_2 we would get this from the pdf by integrating over all other co-ordinates namely

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, x_3, \ldots, x_n) dx_1 dx_3 \cdots dx_n.$$

Definitions of expectations of functions of random variables analogous to the 2 variable setup.

Example of Jointly distributed random variables

Multinomial distribution

Setting:

- Conducting a sequence of n independent trials.(e.g. picking $n = 2000$ random voters)
- Each trial has exactly r possible outcomes (e.g. $r = 3$, Democrat, Republican or Independent).
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  - P(trial = outcome 1) $= p_1$
  - P(trial = outcome 2) $= p_2$
  - $\dots$
  - P(trial = outcome r) $= p_r$ Obviously $\sum_{i=1}^{r} p_r = 1$.
- These probabilities remain the same from trial to trial.

Random variables of interest

- $(X_1, X_2, \dots, X_r)$ where
- $X_1 =$ # of trials which resulted in outcome 1
- $X_2 =$ # of trials which resulted in outcome 2
- $\dots$
- $X_r =$ # of trials which resulted in outcome r.
- Note: $X_1 + X_2 + \cdots X_r =$

Multinomial probability mass function

Joint pmf for Multinomial

Fix any set of integers $n_1, n_2, \dots n_r$ with $0 \le n_i \le n$ and $\sum_{i=1}^r n_i = n$. Then

$$\begin{aligned} p_{X_1,\dots,X_r}(n_1, n_2, \dots, n_r) &= P(X_1 = n_1, X_2 = n_2, \dots, X_r = n_r) \\ &= \frac{n!}{n_1! n_2!, \dots n_r!} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}. \end{aligned}$$

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Multinomial distribution example

Example

In the 2000 election the percentage of vote for Bush/Gore/other was $.48, .48$ and $.04$. Suppose that you sample a 3000 voters from this population (with replacement) and ask who they voted for. If X_B, X_G, X_O denote the number of voters for the various voters, give the distribution of the random vector (X_B, X_G, X_O).

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Jointly distributed random variables: Independence

Independence: Two random variables X and Y are *independent* if, for any sets A and B,

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

(that is, the events $\{X \in A\}$ and $\{Y \in B\}$ are independent). Otherwise, we say that X and Y are *dependent*.

Equivalent condition 1: It can be shown that independence is equivalent to

$$F(a, b) = F_X(a)F_Y(b), \quad \text{all } a, b,$$

where $F(a, b)$ is the joint c.d.f. of X and Y, F_X is the c.d.f. of X and F_Y is the c.d.f. of Y.

Equivalent condition 2: Discrete case: Independence is equivalent to

$$p(x, y) = p_X(x)p_Y(y), \quad \text{all } x, y,$$

where $p(x, y)$ is the joint p.m.f. of X and Y, p_X is the p.m.f. of X and p_Y is the p.m.f. of Y.[1]

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[1] This is also equivalent to $p(x, y) = h(x)g(y)$ for some functions h and g and all x, y.

Equivalent condition 3: Jointly continuous case: Independence is equivalent to

$$f(x,y) = f_X(x)f_Y(y), \quad \text{all } x, y,$$

where $f(x,y)$ is the joint p.d.f. of X and Y, f_X is the p.d.f. of X and f_Y is the p.d.f. of Y.

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Jointly distributed random variables: Independence

Equivalent condition 3 cont'ed: Jointly continuous case: Independence is equivalent to

$$f(x, y) = h(x)g(y), \quad \text{all } x, y,$$

for some functions h and g.

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Jointly distributed random variables: Independence

Was on an exam in the past: An electronic system works until **either** of the two critical components fail at which point the system stops running. The joint density function of the lifetimes of the two components (X, Y) measured in years is

$$f(x, y) = \frac{x + y}{8}, \qquad 0 < x < 2 \text{ and } 0 < y < 2.$$

1. Are X and Y independent?
2. What is the probability that the systems stops working in the first half year of operation?

What if the joint density function is

$$f(x, y) = 24xy, \quad 0 < x < 1, 0 < y < 1, 0 < x + y < 1$$

and is equal to 0 otherwise?

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Solution continued

Problem: If the joint density function of X and Y is

$$f(x, y) = 8e^{-4x}e^{-2y}, \quad 0 < x < \infty,\ 0 < y < \infty$$

and is equal to 0 outside this region, are the random variables independent?

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