STOR 435.001 Lecture 12

Continuous Random Variables - III

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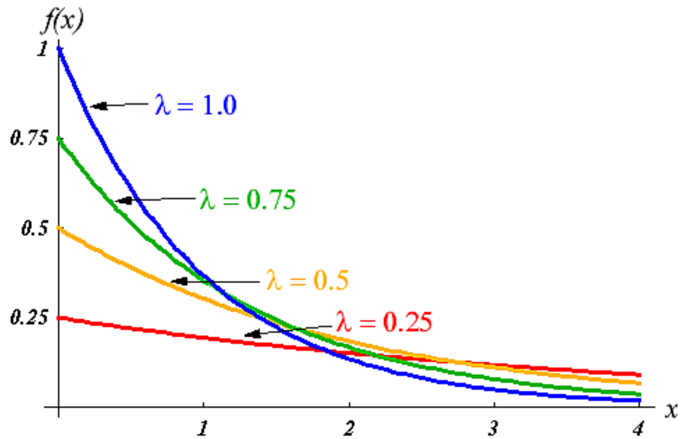
Definition

X is a exponential random variable with parameter $\lambda > 0$ if its density is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x < 0. \end{cases}$$



Notation: $X \sim Exp(\lambda)$.

Continuous random variables: Exponential distribution

If $X \sim Exp(\lambda)$, then:

$F(a) =$

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$EX =$

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$Var(X) =$

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Motivation

Why is this a super important distribution?

- **Interest 1:** In practice, an exponential random variable is often the time until some event (earthquake, phone call, time for next patient to be admitted to the hospital etc) occurs. Recall that this is also the case in the Poisson process context.
- **Interest 2:** Foundation of the basis of things like Queuing theory (which models how systems such as businesses or routers in the internet work). See "Queuing.ppt" in the Extras folder.
- **Interest 3:** Owing to the *lack of memory* property (next slide), forms the foundation of an entire field called *Markov processes*. See `https://en.wikipedia.org/wiki/Markov_chain` for applications of this fundamental model in Physics, Chemistry, Speech recognition, information theory and coding, genetics and even music.
- **Interest 4:** Shows up as the building block of more complicated random variables such as the Gamma random variable (which is one of the main things used in things like text mining and clustering).

Continuous random variables: Exponential distribution

Moment generating function

If $X \sim Exp(\lambda)$ then the MGF $M_X(t) = E(\exp(tX))$ is given by

$$M_X(t) = \begin{cases} & \text{if } t < \lambda \\ \infty & \text{if } t \geq \lambda \end{cases}$$

Memoryless property: $X \sim Exp(\lambda)$ has the following *memoryless* property.

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Fact: IF X is a positive continuous random variable with memoryless property then X has exponential distribution

Why?

- Let F be the cdf of X (so a **increasing function**) and let $G(t) = 1 - F(t) = P(X > t)$. Here $G(\cdot)$ is **decreasing function**
- Lack of memory property means for any s and t,

$$G(s+t) = G(s)G(t).$$

- Differentiate with respect to s (with t fixed) to get

$$G'(s+t) = G(t)G'(s)$$

With $s = 0$, one gets the differential equation

$$G'(t) = G'(0)G(t)$$

- Since G is a decreasing function, $G'(0) = c < 0$. Let $\lambda = -c$.
- The above differential equation looks like

$$\frac{d}{dt}G(t) = cG(t).$$

The solution of this is

$$G(t) = Ae^{ct} = Ae^{-\lambda t}.$$

Here $A = G(0) = P(X > 0) = 1$.
- So

$$F(t) = 1 - G(t) = 1 - e^{-\lambda t}.$$

Connection to Poisson process

- Suppose $(N(t) : t \geq 0)$ is a Poisson process (see Lecture 7) with rate λ. To keep things concrete suppose $N(t)$ counts the number of earthquakes (of size above 4 say on the Richter scale) in the interval $[0, t]$ in some location in California. Here 0 is starting from NOW.
- Let X be the time of the first earthquake. Fact: under the above assumption, X has an exponential distribution with parameter λ

Example: Suppose that the number of hours that a computer hard drive can run before it conks off is exponentially distributed with an average value of 43,800 hours (5 years). If Jan has had the laptop for three years and is now planning to go on a 6 month (4380 hours) trip around the world with his laptop. What is the probability Jan can go on the trip without having to replace the hard drive during the trip? What can be said when the distribution is not exponential?

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Definition

X is a gamma random variable with parameters $\lambda > 0$ and $\alpha > 0$ if its density is

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & x > 0, \\ \\ 0, & x < 0, \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy$$

is the so-called **gamma function**.

Notation: $X \sim \Gamma(\alpha, \lambda)$. Interest: $\Gamma(n, \lambda) =$ amount of time to wait till n events occur (in the Poisson context)

Gamma distribution: Motivation

- Another super important distribution in applications.
- How many of you read news on sites such as Google news? `https://news.google.com/`
- Have you ever wondered how news articles automatically get sorted into different topics?
- Do you think there are people manually sorting these?
- Amazingly powerful machine learning technique called Latent Dirichlet Allocation (LDA).
- Gamma distribution: building block of the above scheme

Continuous random variables: Gamma distribution

Properties of the Gamma function

1. Can be calculated explicitly mainly only for integers: For an integer $n \geq 1$

$$\Gamma(n) = (n-1)!$$

2. For any $\alpha > 0$,

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$$

Why? ☼

Properties of the Gamma distribution

1. $E(X) = \alpha/\lambda$ ☼

2. Var$(X) = \alpha/\lambda^2$ ☼

Continuous random variables: Gamma distribution

Properties of the Gamma distribution 2

Moment generating function:

$$M_X(t) = \begin{cases} & \text{if } t < \lambda \\ \infty & \text{if } t \geq \lambda \end{cases}$$

Why is the Gamma distribution important?

Later we will see that if $X_1, X_2, \ldots, X_n$ are independent random variables each with $Exp(\lambda)$ distribution then $S_n = X_1 + \cdots X_n$ has a Gamma distribution with parameters λ and $\alpha = n$.

Other distributions

Lots of other distributions, of great importance in applications but **not covered** in this course.

1. **Weibull distribution:** Arises widely in engineering problems.
2. **Beta distribution:** Extensions of these major building blocks of *Topic modeling* (How does Google classify news articles automatically into different topics?)
3. **Lognormal distribution:** Finance, Black Scholes etc.

Functions of random variables

Setting

1. Often interested in distributions of functions of random variables.
2. Especially important in generating random variables on the computer.
3. Example: Lots of algorithms to generate Uniform random number U in the interval $(0, 1)$. Now suppose we want to generate $X = Exp(1)$. How to do so?
4. Fact: If U is Uniform $(0, 1)$ then $X = -\log_e(U)$ has $Exp(1)$ distribution.

Three steps

Given distribution of random variable X and a function g want to find pdf of $Y = g(X)$.

1. Figure out the range of the random variable Y.
2. Calculate cdf of Y: $F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$.
3. Calculate pdf of Y: $f_Y(y) = \frac{d}{dy} F_Y(y)$.

Functions of random variables

Example 1

If X is a uniform $(0,1)$ r.v and $n \geq 1$ is a fixed integer calculate the pdf of $Y = X^n$.

Example 2

If X is a uniform $(0,1)$ r.v then calculate the pdf of $\sqrt{X}$.

Example 3

Suppose X has a general pdf f, calculate pdf of X^2.

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