STOR 435.001 Lecture 10

Continuous Random Variables - I

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Continuous random variables

Continuous random variables: set of possible values is uncountable, e.g. $(-\infty, \infty)$, $(0, \infty)$, $(0, 1)$, etc. Examples: time till next earthquake, height of a randomly selected person, length of time we spend on phone for customer service.

- Contable vs. uncountable

Advantages of continuum over discrete: mathematical model that often is easier to use.

Assigning probabilities: Note that we cannot assign probabilities to each value. Why not? How to assign probabilities then?

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Approximation of discrete by continuum: We want to be able to think of continuum as an approximation of discrete. How should we assign probabilities then?

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Definition

X is a continuous random variable if there is a non-negative function f on $(-\infty, \infty)$ such that

$$P(X \in B) = \int_B f(x)dx$$

for any set B of $(-\infty, \infty)$. The function f is called the probability density function of X.

In particular:[1]

$$1 = P(X \in (-\infty, \infty)) = \int_{-\infty}^{\infty} f(x)dx$$

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

$$P(X = a) = \int_{\{a\}} f(x)dx = 0$$

[1] Not surprisingly, we will also start to rely heavily on calculus from now on.

Cumulative distribution function:

$$F(a) = P(X \leq a) = P(X < a) = \int_{-\infty}^{a} f(x)dx$$

that is, the c.d.f. F is the integral of the density f. Note that F is a continuous function (even if f is not).

Note:

$$F'(a) = \frac{dF}{da}(a) = f(a)$$

that is, the density f is the derivative of the c.d.f. F.

Important perspective: Note that, for small ϵ,

$$P\left(a - \frac{\epsilon}{2} \leq X \leq a + \frac{\epsilon}{2}\right) = \int_{a-\frac{\epsilon}{2}}^{a+\frac{\epsilon}{2}} f(x)dx \approx f(a)\,\epsilon$$

if f is continuous at $x = a$. In other words, $f(a)$ is a measure of how likely X will be near a.

Note: The above calculation also says that for a continuous random variable, for any fixed number a, the probability the random variable takes the value exactly equal to a, namely $\mathbb{P}(X = a) = 0$!

Example: Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} C(4 - x^2), & -2 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the value of C? (b) Find $P(X > 1)$.

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In the context of the above problem: (c) Find $\mathbb{P}(1 < X < 2.5)$ (d) Find $\mathbb{P}(X = 1)$.

Example: For a particular stat exam, the amount of the time to complete in hours is known to have density

$$f(x) = 1, \qquad 0 < x < 1$$

Let the random variable $Y = e^X$. What is the pdf of Y?

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Definition

If X is a continuous random variable with density f, its expected value (or mean) is defined as

$$EX = \int_{-\infty}^{\infty} xf(x)dx.$$

Compare with: If X is a discrete random variable with p.m.f. $p(x)$, its expected value is defined as

$$EX = \sum xp(x).$$

Computing expectation of function of continuous random variable

If X is a continuous random variable with density f and g is a function, then

$$Eg(X) = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

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Example

Jan has two nieces (Vivian and Sofia) who are fighting over a chocolate bar. Jan decides to split the chocolate bar for the nieces. He splits it at a uniform location U on the bar namely the density of U is given by

$$f_U(u) = 1, \qquad 0 < u < 1$$

Jan fixes a number p in $(0, 1)$ and tells Vivian she will get the portion of the bar that contains p. Find the expected length of the piece that Vivian gets.

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Example 2d: Suppose that if you are s minutes early for an appointment, then you incur the cost cs, and if you are s minutes late, then you incur the cost ks. Suppose also that the travel time from where you presently are to the location of your appointment is a continuous random variable having probability density function f. Determine the time at which you should depart if you want to minimize your expected cost.

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A corollary: If a and b are constants, then

$$E(aX+b) = aEX + b$$

Variance:

$$Var(X) = E(X-EX)^2 = EX^2 - (EX)^2$$

where

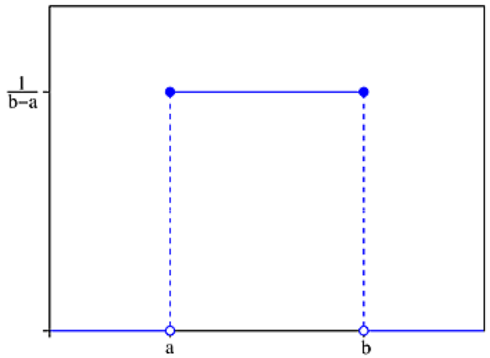
$$EX^2 = \int_{-\infty}^{\infty} x^2 f(x)dx$$

Standard deviation: $\sqrt{Var(X)}$

Definition

X is a uniform random variable on (a, b) if its density is

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$



Notation: $X \sim U(a, b)$. Note: $P(x_1 < X < x_2) = P(x_1 + h < X < x_2 + h)$ for $x_1, x_2, x_1 + h, x_2 + h \in (a, b)$.

Continuous random variables

If $X \sim U(a, b)$, then:

$EX =$

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$Var(X) =$

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Problem 12: A bus travels between the two cities A and B, which are 100 miles apart. If the bus has a breakdown, the distance from the breakdown to city A has a uniform distribution over (0, 100). There is a bus service station in city A, in B, and in the center of the route between A and B. It is suggested that it would be more efficient to have the three stations located 25, 50, and 75 miles, respectively, from A. Do you agree? Why?

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Solution continued