STOR 435 Lecture 5

Conditional Probability and Independence - I

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Motivation

Basic point

- Think of probability as the amount of belief we have in a particular outcome.
- If we are rational, given data we update our prior beliefs in light of this data.

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- **Extreme Example:**
  1. Friend fills up a box with 10 balls of two type: Red and White.
  2. They flipped a coin and either put in 2 Red and 8 Black or 8 Red and 2 black.
  3. Since coined flipped – chance $1/2$ and $1/2$ for either
  4. You pick three balls **without replacement** and find all 3 of them are red.

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  5. What is your best guess on what is in the urn? How certain are you?

Information updating

$$\text{Use: } \underbrace{\text{Initial Belief}}_{\text{Prior probability}} \text{ and } \text{Observed Data} \Longrightarrow \underbrace{\text{New improved Belief}}_{\text{Posterior Probability}}$$

Major road block

1. What to do when no initial belief exists?
2. LaPlace suggested treating everything equally likely. Fisher pointed out that knowing things are equally likely and not knowing are very different states of nature.

History time

Bayes theorem and conditional probability

- Many useful applications
  1. Alan Turing and cracking of the “ENIGMA” (German navy secret code).
  2. Cold War: Finding missing H-bomb, nuclear subs etc.
  3. Bayesian spam filters, computer science, Google, neuroscience (adaptive brains).

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Origins

- Name goes back to Reverend Thomas Bayes in the 1740s.
  - Amateur mathematician (owing to religious persecution was barred from English universities).
- Discovered independently and developed in a tremendous number of applications by Laplace, the “Newton of France”.

History

Figure: “Thomas Bayes”. Licensed under Public domain via Wikimedia Commons

- Bayes never actually published his results.
- When he died, relatives asked another famous amateur mathematician (Price) to look at his mathematical work. He cleaned it up and sent for publication.
- The form we use formulated by Laplace one of the most powerful mathematicians in the history of science.
- Laplace made fundamental contributions in a myriad of sciences including Statistics, celestial mechanics, physics, biology and earth sciences.
- *Laplace* in this course: **Central limit theorem; Normal approximation to the Binomial**

Conditional Probability: Motivating Problem

Example 1

Jan is teaching a large Statistics course. The composition of his course is as follows:

	In State	Out State	Total
Female	52	18	70
Male	15	15	30
Total	67	33	100

You are planning on taking the course the next semester and so ask Jan to pick one the students in his class at random so that you can talk to him/her. Calculate probability of the following

1. Chosen student is In State.
2. Chosen student is Female.
3. Jan tells you that the chosen student is female. What is the chance that the student is also In State?
4. Jan tells you that the chosen student is male. What is the chance that the student is also In State?

Solution to Example 1

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Definition of conditional probability

Define: probability that E occurs given that F has occurred. Name: conditional probability of E given F. Notation: $P(E|F)$.

Examples

Ex. 1: toss 2 dice; $E = \{\text{first die is } 3\}$ and $F = \{\text{sum is } 8\}$. Find $P(E|F)$.

Ex. 2: blood test for a disease; false positive can happen; $E = \{\text{person has disease}\}$ and $F = \{\text{test is positive}\}$. Find $P(E|F)$.

Conditional Probability and Independence

Definition

If $P(F) > 0$,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

Note 1

through Venn diagrams:

Note 2

For equally likely outcomes,

$$P(E|F) = \frac{\frac{\text{Number of outcomes in } E \cap F}{\text{Total number of outcomes in } S}}{\frac{\text{Number of outcomes in } F}{\text{Total number of outcomes in } S}} = \frac{\text{Number of outcomes in } E \cap F}{\text{Total number of outcomes in } F},$$

so that one can think of F as a new (reduced) sample space.

Example

Jane has 2 kids. Note that the sample space is $\{BB, BG, GB, GG\}$. You may assume that all possible outcomes are equally likely. You are told that one of the kids is a girl. What is the chance that the other one is also a girl?

Think-Pair-Share

Example

Jane has 2 kids. Note that the sample space is $\{BB, BG, GB, GG\}$. You may assume that all possible outcomes are equally likely. You are told that one of the kids is a girl. What is the chance that the other one is also a girl?

- A 1/3
- B 1/2
- C 2/3
- D Something else

Example 2

Two cards are randomly chosen without replacement from an ordinary deck of 52 cards. Let B be the event that both cards are hearts and let A be the event that at least one heart is chosen. Find $P(B|A)$.

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Computing probabilities via conditioning

Note: $P(E \cap F) = P(F)\frac{P(E \cap F)}{P(F)} = P(F)P(E|F)$.

More generally:

The multiplication rule

$$P(E_1 \cap E_2 \cap \ldots \cap E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2) \ldots P(E_n|E_1 \cap \ldots \cap E_{n-1})$$

Example 3:

Suppose Jan has a class of 50 students. Out of these, 20 students favorite subject in the whole world is probability and the remaining 30 students whose favorite subject is Combinatorics. Suppose Jan picks 2 students without replacement. If we assume that at each draw each student not yet picked is equally likely to be chosen, what is the probability that both students chosen love probability?

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Example

A recent college graduate is planning to write her actuarial exams. She will write the first exam in June (which she has a .9 probability of passing) and then will write the second exam in July. Given that she passed the 1st exam, the probability that she passes the second exam is .8. What is the probability that she passes both exams?

Example

Klára is interested in pursuing Actuarial Science and wants to talk to an MDS student currently pursing this as a career. There are two STOR 435 Sections (01 and 02).

- Next week, she plans to pick one of these sections uniformly at random and then pick a student uniformly at random from the chosen section (since he does not know who is and is not pursing Actuarial Science).
- Suppose Section 01 has 30% MDS Actuarial students and Section 02 has 50 % actuarial students.
- Suppose next week Section 01 has a midterm and Section 02 does not. So if an actuarial student gets selected from Section 01 then the chance Klára gets a reply is 30 %. If an actuarial student gets selected from Section 02 then the chance Klára gets a reply is 80 %.

Assume that if the student Klára emails is **not** an actuarial student then she **gets no** response.

a What is the chance Klára picks Section 01 and gets a response?

b What is the chance Klára gets a response?

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Solution continued