STOR 435.001 Lecture 3

Axioms of Probability - I

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Introduction

- Today will start our first excursions into formal probability.
- First some philosophical musings on perception of uncertainty and mathematical operations in the light of such uncertainty.

Real world motivation: Cognitive biases

- Make it very hard for human brain to reason and make decisions in the light of uncertainty
- People use simple heuristic principles which lead to severe and systematic errors.
- Two ways of think: **I. Fast** (intuitive) and **II. Slow** (reasoning).
- We think we are in state II most of the time (at least in College!) but most of the time we are in state I.

Example: Daniel Kahnemann and Amos Tversky

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in antinuclear demonstrations. Participants were then asked to rank statements by their probability. Among the sentences were the following:
(A) Linda is a bank teller.
(B) Linda is a bank teller and is active in the feminist movement.”
Which do you think is more probable?

Punchline

In thinking or modeling an experiment that involves randomness or uncertainty, we need to be careful. We need to carefully think about the possible outcomes, the kinds of events we might want to calculate probabilities for and then figure out how to calculate these probabilities. This is what we will start the course with.

Axioms of Probability

Performing an experiment: outcome of the experiment uncertain or non-deterministic.

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Sample space of experiments

Set of all possible outcome

Notation: S.

Example 1:

Experiment = determining the sex of a newborn

☼ $S =$

Example 2:

Experiment = flipping 2 coins

☼ $S =$

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Example 3:

Experiment = There are 6 runners named **A, B, C, D, E, F** that you are observing in the NY marathon. An outcome consists of the order in which these 6 runners finished (example ABCDEF means among these 6 runners A came first, then B and so on). Assume no ties.

☼ # of outcomes in your sample space =

Example 4:

Experiment = observe coin flips till first H appears

☼ $S =$

Example 5:

Experiment = measuring lifetime (in hours) of a bulb

☼ $S =$

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Event: any subset of the sample space S; often described in words.

Notation: A, B, C, D, E, etc. (capitals)

Ex. 1: $E = \{\text{boy is born}\} = \{\text{boy}\}$

Ex. 2: $E = \{\text{1st coin is } H\} =$

Ex. 3: $E =$ amongst the 6 runners, B won the race =

Ex. 4: $E =$ number of coin flips to observe a head is at least 3 =

Ex. 5: $E =$ the lifetime of the bulb is more than ($>$) 10 hours =

Terminology: E occurs = outcome of an experiment is in E

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Operation of events
Two events E, F.

Union of events

Union of E and F = event consisting of all outcomes that are either in E, or in F, or in both E and F.

Notation: $E \cup F$. Terminology: $E \cup F$ occurs = E or F occurs.

Example

In Ex. 2:
$E = \{\text{1st coin is } H\} = \{(H,T),(H,H)\}$
$F = \{\text{outcomes of 1st and 2nd coins are different}\} = \{(H,T),(T,H)\}$

$E \cup F =$

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Intersection

Intersection of E and F = event consisting of all outcomes that are both in E **and** F.

Notation: $E \cap F$. Terminology: $E \cap F$ occurs = E and F occur.

Continuing the previous example

☼ $E \cap F =$

Null events

Null event: event that cannot occur. Notation: $\emptyset$.

Terminology: if $E \cap F = \emptyset$, then E and F are called mutually exclusive (disjoint).

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Extensions to finitely many events

Have events $E_1, E_2, \ldots, E_n$.

Union of E_n's, $1 \leq j \leq n$ = event of outcomes that are in E_j for at least one of the $1 \leq j \leq n$
Notation: $\cup_{j=1}^n E_j$

Intersection of E_n's, $n \geq 1$ = event of outcomes that are in E_n for all $1 \leq j \leq n$
Notation: $\cap_{j=1}^n E_j$

Extension to countably many events:

Consider events $E_1, E_2, \ldots$

Union of E_n's, $n \geq 1$ = event of outcomes that are in E_n for at least one $n \geq 1$
Notation: $\cup_{n=1}^\infty E_n$

Intersection of E_n's, $n \geq 1$ = event of outcomes that are in E_n for all $n \geq 1$
Notation: $\cap_{n=1}^\infty E_n$

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Complement

Complement of E = event consisting of all outcomes of S that are not in E.

Notation: E^c. Terminology: E^c occurs = E does not occur. Note: $S^c = \emptyset$.

Example continued

☼ $E^c =$

Comparison

"E is contained in F" = all outcomes in E are also in F.

Notation: $E \subset F$ or $F \supset E$. Terminology: If E occurs, then F occurs as well.

$E = F$ means $E \subset F$ and $F \subset E$.

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Venn diagrams[1]

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[1] John Venn (1834-1923), British philosopher, mathematician

Axioms of Probability: Laws (rules) for operations

(a) **Commutative laws:** $E \cup F = F \cup E$, $E \cap F = F \cap E$

(b) **Associative laws:** $(E \cup F) \cup G = E \cup (F \cup G)$, $(E \cap F) \cap G = E \cap (F \cap G)$

Distributive laws

(i) $(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$

(ii) $(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$

De Morgan's laws[a]

[a] Augustus De Morgan (1806-1871), British mathematician and logician

$$\left(\bigcup_{n=1}^{k} E_n \right)^c = \bigcap_{n=1}^{k} E_n^c, \quad \left(\bigcap_{n=1}^{k} E_n \right)^c = \bigcup_{n=1}^{k} E_n^c$$

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De Morgan's laws in words

☼ Example: $E =$ Anna saved Elsa, $F =$ Sven saved Kristof city.

(i) $E \cup F$ in words =

(ii) $(E \cup F)^c$ in words =

(iii) So $(E \cup F)^c$ =

(iv) Do the same for $(E \cap F)^c$ and show $= E^c \cup F^c$.

Through Venn diagrams

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Counting: Think-Pair-Share

Problem:

3 balls are selected at random without replacement from a box containing 5 black and 2 red balls. In this experiment, one is interested only in the colors and the order of the 3 selected balls. How many outcomes are there?

- (A) 3
- (B) 4
- (C) 7
- (D) 35

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Problem:

3 balls are selected at random without replacement from a box containing 5 black and 2 red balls. In this experiment, one is interested only in the colors and the order of the 3 selected balls.
Describe the sample space S of this experiment. Let A_k, $k = 2, 3$, be the event that the kth ball selected is black. Which outcomes are in A_2? Which outcomes are in A_3^c?
Describe the event $A_2 \cap A_3^c$ in words and determine its outcomes.

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Example:

Let A, B, and C be three events. Find expressions for the events so that, of A, B, and C,

(b) both A and B occur but not C;

(b) exactly two of the events occur;

(c) not more than one of the events occurs;

Axioms of Probability: Think-Pair-Share

Example:

Find a simple expression for $(A \cup B) \cap (A^c \cup B) \cap (A \cup B^c)$

- (A) $A \cup B$
- (B) $A \cap B$
- (C) S
- (D) $\emptyset$