STOR 435.001 Lecture 2

Combinatorics

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Minor admin details

Waitlist

- All questions regarding course registration and waitlist should be directed to Christine Keat (crikeat@email.unc.edu, Room 321 Hanes Hall, 962-2307).
- You are responsible for verifying your recorded scores (homeworks and midterm exams) during the semester. The grades will all be available on the Sakai class site.
- I expect you to attend the lectures.
- The Honor Code will be observed at all times in this course.
- Email jan.hannig@unc.edu if you are still on the waitlist and hope to get into class so I can add you Sakai. Remember class materials are available at http://www.unc.edu/~hannig/STOR435

Combinatorial Analysis

But . . . hold on!

- **Natural question:** Wait! Isn't this a course in probability? Why are we starting with combinatorics?
- Turns out: if this is your first time seeing Probability then initial definitions could be a little tricky. What does one mean by statements like "If you toss a fair six faced dice then $\mathbb{P}(⚅) = 1/6$"?
- I need to show you lots of examples. Easiest examples to describe: Probability models where "all outcomes are equally likely".
- Need to count outcomes.
- Thus Combinatorics!
- After we get a feeling for probability then more advanced models.

Combinatorial Analysis

The basic principle of counting

Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.

Combinatorial Analysis

Example: How many two letter strings are there? (There are 26 letters).

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Example: Same question as above but the same letters are not allowed.

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The generalized basic principle of counting

If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment; and if ..., then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

Combinatorial Analysis

Figure: By Jaycarlcooper (Own work) via Wikimedia Commons

Example: How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

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Example: Same example as above but the same letters and numbers cannot be repeated.

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Combinatorial Analysis

Example*: Jan has a class S of n students which I will denote by: $S = \{1, 2, \dots, n\}$. Jan is thinking of sending a subset of the students to attend a really cool talk. How many different subsets of S are there? (Including the empty set $\emptyset$ which means Jan decides to send no one from the class.)

For example, for $n = 2$, there are 4 possible subsets: $\emptyset$, $\{1, 2\}$, $\{1\}$ and $\{2\}$.

Combinatorial Analysis

Two important situations of counting:

- Permutations
- Combinations

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Permutations: Consider n distinct objects; how many different ordered arrangements, called permutations, of these objects are?
For example, for $n = 2$ objects a, b, there are two arrangements (permutations): ab, ba.

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Notation: $n!$. Name: n factorial.

Combinatorial Analysis

Example Jan has 12 distinct books that he is going to put on his bookshelf. Amongst these books, 5 are mystery novels, 3 are mathematics books, 2 are parenting books, and 1 is a book on the environment and 1 is a book on “how to teach probability so people don’t fall asleep”. Jan wants to arrange the books on a single shelf so all the books of the same type are together on the shelf. How many different arrangements are possible?

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Combinatorial Analysis ⚠

Combinations: Consider n distinct objects; how many different groups, called combinations, of size r ($1 \leq r \leq n$) of these objects can be formed?
For example, for $n = 3$ objects a, b, c and $r = 2$, there are three possible combinations: $\{a, b\}$, $\{b, c\}$, $\{a, c\}$.

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Notation: $\binom{n}{r}$. Name: n choose r. Convention: $\binom{n}{0} = 1$.

Example: The University wants to from a committee on students affairs. From a group of 5 juniors and 7 seniors, how many different committees consisting of 3 juniors and 2 seniors can be formed? Suppose Melissa and John are two of the seniors but had have a messy brake up and so refuse to be on the same committee together?

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Combinatorial Analysis

Example (tricky!): A car company is developing a new electric car. The car has a set of 100 batteries all arranged in a line. Because these batteries are produced en-mass, not necessarily all batteries work. All the defective batteries are indistinguishable from one another and the same for the working batteries. Suppose out of the 100 batteries, 80 work and 20 do not work. The car will work if no two defectives are consecutive. How many linear orderings are there in which no two defectives are consecutive?

Combinatorial Analysis

$\binom{n}{r}$ are also known as binomial coefficients.

The binomial theorem

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

Multinomial coefficients

For integer $n \geq 1$ and integers $n_1, n_2, \ldots, n_r \geq 0$ such that $n_1 + n_2 + \ldots + n_r = n$,

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$$

For example: $\binom{4}{1,2,1} = \frac{4!}{1!2!1!} = 12$

Note: $r = 2$: $\binom{n}{k, n-k} = \binom{n}{k}$ (binomial coefficients)

Combinatorial Analysis ⚠

Interpretation 1: $\binom{n}{n_1, n_2, \ldots, n_r}$ = Number of ways to divide n distinct objects into r distinct groups of sizes $n_1, n_2, \ldots, n_r$ with $n_1 + n_2 + \ldots + n_r = n$.

For example, with $n = 4$ objects a, b, c, d, $r = 3$ groups of sizes $n_1 = 1, n_2 = 1, n_3 = 2$, the $\binom{4}{1,1,2} = 12$ possibilities are:
$\{a\} \quad \{b\} \quad \{c, d\}$
$\{b\} \quad \{a\} \quad \{c, d\}$
etc.

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Combinatorial Analysis

Interpretation 2: $\binom{n}{n_1, n_2, \ldots, n_r}$ = Number of ordered arrangements of n objects of which n_1 are alike, n_2 are alike, ..., n_r are alike.

For example, with $n = 6$ objects $\square, \square, \square, \bigcirc, \triangle, \triangle$ and hence $r = 3$, $n_1 = 3$, $n_2 = 1$, $n_3 = 2$, there are $\binom{6}{3,1,2} = 60$ arrangements:
$\square\ \square\ \square\ \bigcirc\ \triangle\ \triangle$
$\square\ \square\ \bigcirc\ \square\ \triangle\ \triangle$
etc.

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Combinatorial Analysis

Example: (Interpretation 2) How many different letter arrangements can be formed from the letters MISSISSIPPI? (*M -1 ; I -4; S -4; P - 2*)

Example: (Interpretation 1) The game of bridge is played by 4 players. Each player gets dealt 13 cards. How many bridge deals are possible?

Combinatorial Analysis: Think-Pair-Share

Problem:

3 balls are selected at random without replacement from a box containing 5 black and 2 red balls. In this experiment, one is interested only in the colors and the order of the 3 selected balls. How many outcomes are there?

- A 3
- B 4
- C 7
- D 35

Combinatorial Analysis

$\binom{n}{n_1, n_2, \ldots, n_r}$ are called multinomial coefficients.

The multinomial theorem

$$(x_1 + x_2 + \ldots + x_r)^n = \sum_{\substack{(n_1, n_2, \ldots, n_r) : \\ n_1 + n_2 + \ldots + n_r = n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

That is, the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \ldots + n_r = n$.

Combinatorial Analysis: Importance of distinct vs non-distinct groups

Example A

A fire-marshall has 8 firefighters. He wants to split them into 4 groups of 2 each with **(a)** Group 1 manning the truck **(b)** Group 2 manning the phones **(c)** Group 3 cleaning the station **(d)** Group 4 on standby. How many ways are there to do this?

Example B

A fire-marshall has 8 firefighters. He wants to split them into 4 groups of 2 each. How many ways are there to do this?