STOR 435 – February 16, 2009 Name:________________________

Midterm Exam 2

All problem parts have equal weight. In budgeting your time expect that some problems will take longer than others.

The exam is closed book and closed notes. You will only need a calculator, a pen or pencil, and one sheet of paper with class material that was prepared by you and does not contain any solutions to homework and old exams. If you need additional blank paper for solutions, or work, please ask the instructor. To receive any (or partial) credit for a question, you must show your work. Please give your answers in the space provided. All numerical answers should be expressed as decimal numbers, rather than fractions. When you have finished, please sign the following Honor Code pledge:

I pledge that I have neither given nor received unauthorized assistance during this examination.

Signed: ________________________

1. A biased coin (with probability $P(H) = 0.4$) is tossed 20 times. Let X denote the number of Heads.

   (a) Find $P(X = 2)$.

   (b) Find EX.

   Solution: $X \sim$ Binomial(20,0.4) and therefore $P(X = 2) = \binom{20}{2}(0.4)^2(0.6)^1 = 0.003$ and $EX = 20(0.4) = 8$.

2. Suppose that the distribution function of a random variable X is given by

$$F_X(s) = \begin{cases} 0 & s < -1, \\ 0.1 & -1 \le s < 0, \\ 0.4 & 0 \le s < 1, \\ 1 & 1 \le s. \end{cases}$$

(a) Find the probability mass function $p_X(s)$.
Note: if you cannot do this part you can ask your instructor for the answer so that you can continue with the question. However, points will be deducted — ☐ check if answer is given.

(b) Find EX and $\operatorname{var} X$.

Solution:

(a) By looking at the jumps

$$p_X(s) = \begin{cases} 0.1 & s = -1, \\ 0.3 & s = 0, \\ 0.6 & s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) $EX = (-1)(0.1) + 0(0.3) + 1(0.6) = 0.5$,
$EX^2 = (-1)^2(0.1) + 0^2(0.3) + 1^2(0.6) = 0.7$,
$\operatorname{var} X = 0.7 - (0.5)^2 = 0.45$.

3. Suppose that the distribution function of a random variable X is given by

$$F_X(s) = \begin{cases} 0 & s < -1, \\ \frac{s^3+1}{2} & -1 \le s < 1, \\ 1 & 1 \le s. \end{cases}$$

   (a) Find the density $f_X(s)$.
   Note: if you cannot do this part you can ask your instructor for the answer so that you can continue with the question. However, points will be deducted — ☐ check if answer is given.

   (b) Find EX and $\operatorname{var} X$.

Solution:

(a) By taking a derivative

$$f_X(s) = \begin{cases} \frac{3}{2}s^2 & -1 < s < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) $EX = \int_1^1 s \, \frac{3}{2}s^2 \, ds = 0$,
$EX^2 = \int_1^1 s^2 \, \frac{3}{2}s^2 \, ds = \frac{3}{5}$,
$\operatorname{var} X = \frac{3}{4} - 0^2 = 0.6.$

4. Assume that X is a continuous random variable with density

$$f_X(z) = \begin{cases} ce^z & -1 < z < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find a constant c that makes $f_X(z)$ a density. (Use this constant in the following parts.)

(b) Is the random variable X^2 continuous or discrete? If it is continuous, find its density. If it is discrete find its mass function.

(c) Define $Y = \operatorname{sign} X$, i.e., $Y = -1$ if $X < 0$, $Y = 0$ if $X = 0$ and $Y = 1$ if $X > 0$. Is Y continuous or discrete? If it is continuous, find its density. If it is discrete find its mass function.

Solution:

(a) $c^{-1} = \int_{-1}^{1} e^z \, dz = e - e^{-1}$ and $c = \frac{1}{e+e^{-1}}$.

(b) Recall that in general

$$F_{X^2}(s) = P(X^2 \le s) = \begin{cases} 0 & s < 0 \\ P(-\sqrt{s} \le X \le \sqrt{s}) = F_X(\sqrt{s}) - F_X(\sqrt{s}_-) & s \ge 0. \end{cases}$$

By taking a derivative

$$f_{X^2}(s) = \begin{cases} \frac{f_X(\sqrt{s}) + f_X(-\sqrt{s})}{2\sqrt{s}} & s > 0 \\ 0 & s \le 0. \end{cases}$$

In our case we have continuous random variable with density

$$f_{X^2}(s) = \begin{cases} \frac{e^{\sqrt{s}} + e^{-\sqrt{s}}}{2\sqrt{s}(e+e^{-1})} & 0 < s < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) Y takes only 3 values $-1, 0, 1$ and therefore it is discrete.
$P(Y = -1) = P(X < 0) = \int_{-1}^{0} \frac{e^z}{e-e^{-1}} \, dz = \frac{1-e^{-1}}{e-e^{-1}}$,
$P(Y = 0) = P(X = 0) = 0$,
$P(Y = 1) = P(X > 0) = \int_{0}^{1} \frac{e^z}{e-e^{-1}} \, dz = \frac{e-1}{e-e^{-1}}$.

5. If $EX = 1$ and $\operatorname{var} X = 5$ find

   (a) $E[(2+X)^2]$,
   (b) $\operatorname{var}(4+3X)$.

   Solution:

   (a) $EX^2 = \operatorname{var} X + (EX)^2 = 5 + 1^2 = 6$ and
   $E[(2+X)^2] = 4 + 4EX + EX^2 = 4 + 4 + 6 = 14$.
   (b) $\operatorname{var}(4+3X) = 3^2 \operatorname{var} X = 45$.

6. Jan's naughty dog Shtutzy does something bad, like eating Jan's food or breaking out of a cage, on average about once a month (the bad events are arriving over time).

   (a) What is the probability that she does nothing bad for 4 months?

   (b) Jan will get really mad at her the third time she does something bad. Let X be the time until Jan get's really mad. What is the distribution of X?

   Note: if you cannot do this part you can ask your instructor for the answer so that you can continue with the question. However, points will be deducted — ☐ check if answer is given.

   (c) Find $EX, \operatorname{var} X, EX^3$.

Solution:

(a) If Y is the number of events in 4 units of time (with the average of 1 event per unit of time), $Y \sim \text{Poisson}(4)$ and $P(Y = 0) = e^{-4} = 0.018$.

(b) The waiting time for the third event has $\Gamma(3, 1)$ distribution.

(c) $EX = \frac{3}{1} = 3$, $\operatorname{var} X = \frac{3}{1^2} = 3$, $EX^3 = \int_0^\infty x^3 \frac{x^2 e^{-x}}{\Gamma(3)} = \frac{\Gamma(6)}{\Gamma(3)} = 60$.