STOR 435 – February 16, 2009 Name:________________________

Old Midterm Exam 1 - Key

All problem parts have equal weight. In budgeting your time expect that some problems will take longer than others.

The exam is closed book and closed note. You will only need a calculator and a pen or pencil. If you need additional paper for solutions, or work, please ask the instructor. To receive any (or partial) credit for a question, you must show your work. Please give your answers in the space provided. All numerical answers should be expressed as decimal numbers, rather than fractions. When you have finished, please sign the following Honor Code pledge. I pledge that I have neither given nor received unauthorized assistance during this examination.

Signed: ________________________

1. Let us assume that the events $A_1, \ldots, A_k$ are independent. Prove or disprove $P(\cup_{l=1}^{k} A_l) = 1 - \prod_{l=1}^{k}(1 - P(A_l))$.

   Solution:

$$
\begin{aligned}
P(\cup_{l=1}^{k} A_l) &= 1 - P((\cup_{l=1}^{k} A_l)^{\mathsf{C}}) \\
&= 1 - P(\cap_{l=1}^{k} A_l^{\mathsf{C}}) \\
&= 1 - \prod_{l=1}^{k} P(A_l^{\mathsf{C}}) \\
&= 1 - \prod_{l=1}^{k} (1 - P(A_l))
\end{aligned}
$$

   The second equality is due to the De-Morgan's laws and the third due to independence.

2. A fair die is tossed n times. What is the probability of the event at east one "face" never appears? **Solution:**

   Denote A_k the evenet that "the face k does not appear". We need to calculate $P(A_1 \cup \cdots \cup A_6)$. Using the inclusion and exclusion principle and symmetry we get:

$$\begin{aligned} P(\bigcup_{j=1}^{6} A_j) &= \sum_{i=1}^{6} P(A_i) - \sum_{1 \le i < j \le 6}^{6} P(A_i \cap A_j) + \sum_{1 \le i < j < k \le 6}^{6} P(A_i \cap A_j \cap A_k) - \cdots \\ &= \binom{6}{1}\frac{5^n}{6^n} - \binom{6}{2}\frac{4^n}{6^n} + \binom{6}{3}\frac{3^n}{6^n} - \binom{6}{4}\frac{2^n}{6^n} + \binom{6}{5}\frac{1}{6^n}. \end{aligned}$$

3. Jan's big brown dog Shtutzy knows how to open the fridge. One day Jan leaves a dozen (12) eggs in his fridge. Two of the eggs are rotten, the rest is good. When Jan comes home the fridge is ransacked. Among other things Shtutzy ate 5 eggs out of the dozen. Assume that she picked the eggs at random without paying any attention to whether the eggs are good or not.

   (a) What is the probability that Shtutzy ate exactly one rotten egg?

   (b) Shtutzy is known to have a "stomach of steel". If she eats one rotten egg she will be sick with probability 0.2. If she eats two rotten eggs she will be sick with probability 0.5. If she does not eat any rotten egg she still might be sick from eating too much with probability 0.01. What is the chance Shtutzy will be sick?

   (c) I have a good news, Shtutzy was not sick. Jan wanted to have eggs next morning for breakfast. Since he was sleepy, he picked three out of the remaining eggs at random. What is the chance none of the three eggs is rotten? (Hint: This is a conditional probability!)

Solution: We will denote the following events by: $R_i =$"Shtutzy ate i rotten eggs", $i = 0, 1, 2$, $S =$"Shtutzy got sick", $J =$"Jan had no rotten egg".

(a) $P(R_1) = \frac{\binom{2}{1}\binom{10}{4}}{\binom{12}{5}} = .530.$

(b)

$$\begin{aligned} P(S) &= P(S|R_0)P(R_0) + P(S|R_1)P(R_1) + P(S|R_2)P(R_2) \\ &= 0.01\,\frac{\binom{2}{0}\binom{10}{5}}{\binom{12}{5}} + 0.2\,\frac{\binom{2}{1}\binom{10}{4}}{\binom{12}{5}} + 0.5\,\frac{\binom{2}{2}\binom{10}{3}}{\binom{12}{5}} = 0.185 \end{aligned}$$

(c)

$$
\begin{aligned}
P(J|S^{\complement}) =& \frac{P(J \cap S^{\complement})}{P(S^{\complement})} \\
=& \frac{P(J|S^{\complement} \cap R_0)P(S^{\complement}|R_0)P(R_0)}{1 - P(S)} + \frac{P(J|S^{\complement} \cap R_1)P(S^{\complement}|R_1)P(R_1)}{1 - P(S)} \\
& + \frac{P(J|S^{\complement} \cap R_2)P(S^{\complement}|R_2)P(R_2)}{1 - P(S)} \\
=& \frac{\frac{\binom{5}{3}}{\binom{7}{3}}(0.99)\frac{\binom{2}{0}\binom{10}{5}}{\binom{12}{5}} + \frac{\binom{6}{3}}{\binom{7}{3}}(0.8)\frac{\binom{2}{1}\binom{10}{4}}{\binom{12}{5}} + + \frac{\binom{7}{3}}{\binom{7}{3}}(0.5)\frac{\binom{2}{2}\binom{10}{3}}{\binom{12}{5}}}{1 - 0.185} = 0.501
\end{aligned}
$$

4. Let X be a random variable with probability mass functions

x	-1	0	2
p(x)	1/4	1/2	1/4

Find the following:

(a) EX

(b) Var(X)

(c) E 2^X

(d) P($X^3 \leq 1$)

Solution:

(a) EX $= (-1)\,\frac{1}{4} + 0\,\frac{1}{2} + 2\,\frac{1}{4} = \frac{1}{4}$

(b) Var(X) $= EX^2 - (EX)^2 = (-1)^2\frac{1}{4} + 0^2\frac{1}{2} + 2^2\frac{1}{4} - (\frac{1}{4})^2 = \frac{19}{16}$

(c) E $2^X = 2^{-1}\frac{1}{4} + 2^0\frac{1}{2} + 2^2\frac{1}{4} = \frac{13}{8}$

(d) P($X^3 \leq 1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$