STOR 435 – May 6, 2009 Name:________________________

Final Exam - Quick Key

All problem parts have equal weight. In budgeting your time expect that some problems will take longer than others.

The exam is closed book and closed notes. You will only need a calculator, a pen or pencil, and one sheet of paper with class material that was prepared by you and does not contain any solutions to homework and old exams. If you need additional blank paper for solutions, or work, please ask the instructor. To receive any (or partial) credit for a question, you must show your work. Please give your answers in the space provided. All numerical answers should be expressed as decimal numbers, rather than fractions. When you have finished, please sign the following Honor Code pledge:

I pledge that I have neither given nor received unauthorized assistance during this examination.

Signed: ________________________

1. You are dealt 5 cards from a regular deck of cards.

   (a) What is the probability that all your cards are of the same color? (Recall there is 26 red and 26 black cards in the deck).

   (b) What is the probability that all your cards are of the same color given that you have exactly two aces?

Solution:

(a) $\frac{2\binom{26}{5}}{\binom{52}{5}} = 0.051$

(b) $\frac{2\binom{2}{2}\binom{24}{3}}{\binom{4}{2}\binom{48}{3}} = 0.039.$

2. Joint density of X and Y, denoted by $f_{XY}(x, y)$, is given below. In all three cases find the constant c and decide whether X and Y are independent? Give reasons.

   (a) $f_{XY}(x, y) = c(x + y), \quad 0 < x < 1,\ 0 < y < 2;$

   (b) $f_{XY}(x, y) = c(x + xy), \quad 0 < x < 1,\ 0 < y < 2;$

   (c) $f_{XY}(x, y) = cxy, \quad 0 < x < 1,\ 0 < y < 2x.$

   Solution:

   (a) $c = \frac{1}{3}$, not independent (the joint density is not the product of marginal densities);

   (b) $c = \frac{1}{2}$, independent (the joint density can be factored as a product of $x(1 + y)$);

   (c) $c = 2$, not independent (the joint density is not the product of marginal densities, notice that the bound on y depends on x).

3. Suppose that the distribution function of a random variable X is given by

$$F_X(s) = \begin{cases} 0 & s < -1, \\ 0.2 & -1 \le s < 0, \\ 0.8 & 0 \le s < 1, \\ 1 & 1 \le s. \end{cases}$$

(a) Find the probability mass function $p_X(s)$.

(b) Find EX and $\operatorname{Var} X$.

Solution:

(a)

x	-1	0	1
$p_X(x)$	0.2	0.6	0.2

(b) $EX = (-1)(0.2) + 0(0.6) + 1(0.2) = 0,$
$EX^2 = (-1)^2(0.2) + 0^2(0.6) + 1^2(0.2) = 0.4,$
$\operatorname{Var} X = 0.4 - 0^2 = 0.4.$

4. Let the condition distribution of $Y|X = x$ be $\mathrm{N}(x, 1)$ and $X \sim \mathrm{N}(0, 1)$.

(a) Find EY, $\operatorname{Var} Y$ and $\rho(X, Y)$.

(b) What is the conditional density $f_{X|Y=y}(x)$? Is this a distribution you recognize?

(c) What is the distribution of X^2?

(d) Find the moment generating function of X^2.

Solution:

(a) $EX = 0, \operatorname{Var} X = 1$.
$EY = \int_{-\infty}^{\infty} E[Y|X = x] f_X(x)\, dx = \int_{-\infty}^{\infty} x f_X(x)\, dx = EX = 0,$
$EY^2 = \int_{-\infty}^{\infty} E[Y^2|X = x] f_X(x)\, dx = \int_{-\infty}^{\infty} (x^2 + 1) f_X(x)\, dx$
$= EX^2 + 1 = 1 + 1 = 2,$
$EY^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{Y|X=x}(y) f_X(x)\, dy\, dx = \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} y f_{Y|X=x}(y)\, dy\right) f_X(x)\, dx$
$= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = 1,$
$\rho(X, Y) = \frac{1-0}{\sqrt{2}}$

(b) $f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X=x}(y) f_X(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{(y-x)^2}{2} - \frac{x^2}{2}}\, dx = \frac{e^{-\frac{y^2}{4}}}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \frac{e^{-(x-y/2)^2}}{\sqrt{\pi}}\, dx$
$= \frac{e^{-\frac{y^2}{4}}}{\sqrt{4\pi}} \sim N(0, 2).$

(c) We have shown in class If $X \sim N(0, 1)$ then $X^2 \sim \chi_1^2 = \Gamma(\frac{1}{2}, \frac{1}{2})$.

(d) $M_X^2(t) = \int_0^{\infty} e^{zt} \frac{e^{-z/2}}{\Gamma(1/2)(2z)^{1/2}}\, dz = \frac{1}{(1-2t)^{1/2}} \int_0^{\infty} \frac{(1/2-t)^{1/2} z^{-1/2} e^{-z(1/2-t)}}{\Gamma(1/2)}\, dz$
$= \frac{1}{(1-2t)^{1/2}}.$

5. The annual rainfall (in cm) in a certain area is normally distributed with $\mu = 40$ and $\sigma = 4$. Assume independence between years.

   (a) What is the probability that starting with next year, it will take over 10 years before a year occurs having rainfall over 50 cm?

   (b) What is the probability that the average rainfall in the next 10 years will be between 37 and 43 cm?

   (c) What is the probability that next year rainfall is more than 90% of the rainfall in the following year.

Solution:

(a) $P(\text{"annual rainfall less than 50cm next ten years"}) = P(X < 50)^{10} = P(Z < \frac{50-40}{4})^{10} = 0.9938^{10} = 0.9397$.

(b) $P(37 < \bar{X} < 43) = P(\frac{37-40}{4/\sqrt{10}} < Z < \frac{43-40}{4/\sqrt{10}}) = 0.9821$.?

(c) Notice $X_1 - 0.9X_2 \sim N(40-36, 16+0.9^2\ 16) = N(4, 28.96)$. Thus $P(X_1 > 0.9X_2) = P(X_1 - 0.9X_2 > 0) = P(Z > \frac{0-4}{\sqrt{28.96}}) = 0.7704$.

6. The number of years a certain appliance works is a r.v. with a hazard rate function given by $\lambda(t) = \frac{(1-t)^2+1}{100}$ (time t is measured in years). Find the probability that the appliance will fail within 6 years.
Solution: $P(X < 6) = 1 - e^{-\int_0^6 \lambda(s)\,ds} = 1 - e^{-\frac{48}{100}} = 0.3821.$

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998