Homework set #3
Based on lectures 3 – 4.

1. (a) Prove that if $EX^2 < \infty$ then $P(X - EX \geq t) \leq \frac{\text{Var}X}{\text{Var}X + t^2}$ for all $t > 0$. (Hint: $t \leq E\{|(t - (X - EX))I_{\{X-EX<t\}}\}$ might be useful).

(b) Then show that this inequality cannot be improved. In particular show that for any fixed $t \geq 0$,

$$
\sup_{X} \left( \frac{P(X - EX \geq t)}{\frac{\text{Var}X}{\text{Var}X + t^2}} \right) = 1,
$$

where the supremum goes over all possible random variables satisfying $EX^2 < \infty$.

2. (a) Find the moment generating function of the following distributions. Poisson($\lambda$), Exp($\lambda$) and $\mathcal{N}(0, 1)$

(b) Use the series expansion of the MGF of the standard normal to find the moments $EZ^{2k}$ for $Z \sim \mathcal{N}(0, 1)$.

3. Is it possible for $X, Y, Z$ to have the same distribution and satisfy $X = U(Y + Z)$, where $U \sim U(0, 1)$, and $Y, Z$ are independent of $U$ and each other?

4. Let $X \sim \text{Bin}(n, p)$. Find the MGF of $X$. Show that for $\epsilon > p$

$$
P(X/n \geq \epsilon) \leq \exp \left\{ -n(1-\epsilon) \log \left( \frac{1-\epsilon}{1-p} \right) - n\epsilon \log \left( \frac{\epsilon}{p} \right) \right\}
$$